$\epsilon = 0.1$, the upper bound $\delta = 0.198$ could be concluded, which is incorrect as the ν-metric between P and P_o is actually 1. It is also noted that

$$\begin{aligned}\tilde{G}_o &= \begin{bmatrix}\tilde{M}_o & \tilde{N}_o\end{bmatrix} \\ &= \begin{bmatrix}\frac{1}{a} & \frac{\beta}{a}\end{bmatrix} + \frac{1}{as+b}\begin{bmatrix}\epsilon - \frac{b}{a} & 1 - \frac{\beta b}{a}\end{bmatrix}\end{aligned}$$

with $a = \sqrt{1+\beta^2}$ and $b = \sqrt{1+\epsilon^2}$. Thus for $\alpha = 9$, $\beta = 8$, and $\epsilon = 0.1$, the Hankel norm of $\tilde{G}_o$ is 0.0123, which implies that K exists such that $\mathcal{F}_\ell(G, K)$ is stable, and $\|\mathcal{F}_\ell(G, K)\|_\infty < \gamma$ for any γ such that $1.0001 < \gamma \le \delta^{-1} = 5.0505$, leading to the false conclusion that the true plant P is stabilized by such a controller K. Actually the nominal model as in (13) admits a very large stability margin for the closed-loop system, but unfortunately, P as in (13) is just not in the set of ν-metric uncertainty stabilizable by the same K. In fact, a stable and causal interpolation function Δ_1 can be obtained with $\|\Delta_1\|_\infty \le \delta \approx 2\epsilon/(1+\epsilon^2)$. Moreover $P_1 = F_u(G, \Delta_1)$ is stabilizable by the same controller K. However either P_1 is unstable (violating stability of P), or $\mathrm{wno}(1 + P_o^* P_1) = 0$ [violating continuity and smoothness of $P(j\omega)$]. Thus P_1, constructed using boundary interpolation and linear fraction, is not equivalent to P, which is not stabilized by the same controller. Consequently, the uncertainty model is invalidated in spite of the fact that (11) is satisfied.

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Mean-Square Small Gain Theorem for Stochastic Control: Discrete-Time Case

Jianbo Lu and Robert E. Skelton

***Abstract*—This note presents a small gain theorem in the mean square sense for multiple (interconnected) linear systems with multiplicative noises. The small-gain theorem is proposed in terms of the spectral radius of a matrix, whose elements are the squares of H_2 norms of the involved transfer functions. Both robust stability and performance conditions are characterized by the new small-gain theorem.**

***Index Terms*—Internal mean square stability (IMSS), multiplicative noise uncertainty, robust control, stochastic control.**

I. Introduction

The control problem for linear systems with multiplicative noises has been shown applicable in many engineering problems [3], [6]. However the control synthesis problem, especially for output feedback, has not yet been solved. The approach proposed in [5] might provide some insight for this problem. This motivates the study for more general system descriptions. Existing studies are usually dealing with single linear systems of multichannel multiplicative noises, and we consider the interconnections among multiple linear systems of multiplicative noises. One application of this problem is to design robust control for linear systems of multiplicative noises with respect to uncertainties of the same type, or linear systems with structured white parameters.

Two stability concepts in the mean-square sense have been studied here, namely, the usual *mean-square stability* (MSS) and the *internal mean-square stability* (IMSS). A multiloop small gain condition is established for IMSS, which is also applicable to MSS due to the equivalence of those two stability concepts. The condition is characterized by the spectral radius of a matrix whose elements are the squares of H_2 norms of the involved transfer functions. The spectral radius of certain nonnegative matrix has been pursued before. It has been used to characterize upper bounds for structured singular value [1] and in l^1 optimal control [4]. In [4], the elements of the matrix whose spectral radius is taken are l^∞-induced norms (without squares) of transfer functions.

The note is organized as follows. Section II studies the interconnection between multiple linear systems with multiplicative noises. The mean-square small gain theorem for a single loop interconnection is proposed for deducing IMSS. Section III deals with robustness analysis for linear stochastic systems of norm-bounded dynamic stochastic perturbations. Section IV concludes the note.

The following notations are used in this note. $\mathbb{R}_+$ and $\mathbb{R}$ denote the sets of strictly positive-real numbers and real numbers respectively. The sets of all $n \times n$ matrices whose elements belong to $\mathbb{R}$ and $\mathbb{R}_+$ are denoted as $M(\mathbb{R}^n)$ and $M(\mathbb{R}_+^n)$ respectively. All the strictly positive definite matrices in $M(\mathbb{R}^n)$ consist of a set denoted as $M_+(\mathbb{R}^n)$. I denotes a unit matrix whose dimension can be determined from the context. $\det(\cdot)$, $\overline{\lambda}(\cdot)$, $\rho(\cdot)$, $\mathrm{tr}(\cdot)$ and $(\cdot)^T$ denote the determinant, the largest eigenvalue, the spectral radius, the trace and the transpose of a

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J. Lu is with Research and Vehicle Technology, Ford Motor Company, Dearborn, MI 48124 USA (e-mail: jlu10@ford.com).

R. E. Skelton is with the Department of Mechanical and Aerospace Engineering, University of California at San Diego, La Jolla, CA 92093-0411 USA (e-mail: bobskelton@ucsd.edu).

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matrix $(\cdot)$ respectively. $\otimes$ denotes Kronecker product, and $\mathrm{vec}(\cdot)$ is the column stack of a matrix. $\|\cdot\|$ is the Euclidean norm of a vector. $\mathbf{E}[\cdot]$ denotes the usual expectation operator of a stochastic variable. For a discrete time stochastic process $z = \{z_k\}_{k=0}^{\infty}$

$$\mathbf{E}_{\infty}[z] \triangleq \lim_{k\to\infty} \mathbf{E}[z_k].$$

δ_{ij} is the Kronecker delta function ($\delta_{ij} = 1$ if $i = j$ and zero otherwise). The set $\mathbf{v}$ of all discrete time stochastic processes with finite variances is defined as

$$\mathbf{v} \triangleq \{z = \{z_k\}_{k=0}^{\infty} : \|z\|_{\mathbf{v}} \text{ is finite}\}$$

where the signal norm is defined as $\|z\|_{\mathbf{v}} \triangleq \sqrt{\mathbf{E}_{\infty}[\|z\|^2]}$.

II. Stochastic Models and Mean-Square Small Gain Theorem

Consider a set of stochastic systems, $\boldsymbol{S}$. Each element $\mathcal{T}$ of $\boldsymbol{S}$ is a mapping from the signal space $\mathbf{v}$ to the signal space $\mathbf{v}$, i.e., $\mathcal{T}: \mathbf{v} \mapsto \mathbf{v}$. $\mathcal{T}$ has the following state-space description:

$$x_{k+1} = Ax_k + [Bw_k]v_k, \qquad z_k = Cx_k \tag{1}$$

where $x_k \in \mathbb{R}^{n_x}$ is the system state, $z_k \in \mathbb{R}$ is the system output, $v_k \in \mathbb{R}$ is the system input. Notice that there is a multiplier w_k in the input channel, and $\mathcal{T}$ is said to have multiplicative noise if this multiplier w_k is a white noise process. A linear system associated with $\mathcal{T}$ is denoted as T, whose state-space description is

$$x_{k+1} = Ax_k + Bv_k, \qquad z_k = Cx_k$$

i.e., T comes from $\mathcal{T}$ by dropping the white noise w_k in the input channel. The following assumptions are made for each element $\mathcal{T}$ of $\boldsymbol{S}$.

A1) The multiplier $w = \{w_k\}_{k=0}^{\infty}$ is a scalar white noise process satisfying for all $t \geq 0$ and $k \geq 0$

$$\mathbf{E}[w_k w_t] = \delta_{kt}, \qquad \mathbf{E}[w_k] = 0$$

which is called the *stochastic multiplier of the system* $\mathcal{T}$. The stochastic multipliers of different elements in $\boldsymbol{S}$ are uncorrelated.

A2) The initial state x_0 of $\mathcal{T}$ is a normal random variable which is uncorrelated with w_k and v_k for all $k \geq 0$.

A3) w_k and v_k are uncorrelated with each other for all $k \geq 0$.

A4) A is a discrete-time stable matrix, i.e., $\rho(A) < 1$.

For each element $\mathcal{T}$ in $\boldsymbol{S}$, the system norm of $\mathcal{T}$ is defined by

$$\|\mathcal{T}\|_{\mathbf{s}} \triangleq \max_{\|v\|_{\mathbf{v}}\neq 0} \frac{\|z\|_{\mathbf{v}}}{\|v\|_{\mathbf{v}}}.$$

Under assumptions A1)$\sim$A4), the following result is easy to obtain.

Lemma 2.1: Consider the stochastic system in (1). If $v \in \mathbf{v}$ with $\|v\|_{\mathbf{v}} \neq 0$, then $\|z\|_{\mathbf{v}} = \|\mathcal{T}\|_{\mathbf{s}}\|v\|_{\mathbf{v}}$. Let T be the linear system associated with the stochastic system $\mathcal{T}$, then $\|\mathcal{T}\|_{\mathbf{s}} = \|T\|_2$, where $\|\cdot\|_2$ is the usual H_2 norm of a linear stable system.

Suppose $\mathcal{T}_1, \mathcal{T}_2 \in \boldsymbol{S}$ and consider the interconnection in Fig. 1. Let $\mathcal{T}_i$ be the system of the following state-space description for $i = 1, 2$:

$$\begin{aligned} x_{1_{k+1}} &= A_1 x_{1_k} + [B_1 w_{1_k}] z_{2_k} \qquad & z_{1_k} &= C_1 x_{1_k} \\ x_{2_{k+1}} &= A_2 x_{2_k} + [B_2 w_{2_k}] z_{1_k} \qquad & z_{2_k} &= C_2 x_{2_k} \end{aligned}$$

where $x_i \in \mathbb{R}^{n_{x_i}}$ is the system state and n_{x_i} is the state dimension of $\mathcal{T}_i$. It is not hard to verify that for each $k \geq 0$, if the initial states x_{1_0} is uncorrelated with w_{2_k}, then x_{2_k} is uncorrelated with z_{1_k}; if the initial state x_{2_0} is uncorrelated with w_{1_k}, then x_{1_k} is uncorrelated with z_{2_k}.

The augmented dynamics for the interconnection in Fig. 1 is

$$\hat{x}_{k+1} = \begin{bmatrix} A_1 & B_1 C_2 w_{1_k} \\ B_2 C_1 w_{2_k} & A_2 \end{bmatrix} \hat{x}_k \tag{2}$$

where $\hat{x}_k = [x_{1_k}^T \;\; x_{2_k}^T]^T$ is the augmented system state. While the augmented dynamics for the interconnection in Fig. 2 is

$$\hat{x}_{k+1} = \begin{bmatrix} A_1 & B_1 C_2 w_{1_k} \\ B_2 C_1 w_{2_k} & A_2 \end{bmatrix} \hat{x}_k + \begin{bmatrix} B_1 w_{1_k} & \\ & B_2 w_{2_k} \end{bmatrix} \cdot \begin{bmatrix} d_{2_k} \\ d_{1_k} \end{bmatrix}. \tag{3}$$

Notice that (2) and (3) describe systems with multiplicative noises of some structures, or systems with *structured multiplicative noises,* or systems with *structured white parameter uncertainties.*

Definition 2.2: The interconnection in Fig. 1 is said to be mean-square stable if for any initial condition $\hat{x}_0$, $\lim_{k\to\infty} \mathbf{E}[\hat{x}_k \hat{x}_k^T] = 0$, where $\hat{x}_k$ is the system state of (2). The interconnected system in Fig. 2 is said to be mean-square stable if for any uncorrelated white noise processes d_1 and d_2, $\lim_{k\to\infty} \mathbf{E}[\hat{x}_k \hat{x}_k^T]$ is finite.

It is not hard to prove that the mean-square stability of the system in Fig. 1 is equivalent to the mean-square stability of the system in Fig. 2. Now let's consider the condition for the mean-square stability.

Theorem 2.3: The interconnection in Fig. 1 is mean-square stable iff for any $\alpha_1 > 0$ and $\alpha_2 > 0$ there exists $X_1 \in M_+(\mathbb{R}^{n_{x_1}})$ and $X_2 \in M_+(\mathbb{R}^{n_{x_2}})$ such that

$$\begin{aligned} X_1 &= A_1 X_1 A_1^T + B_1 C_2 X_2 C_2^T B_1^T + \alpha_2 B_1 B_1^T, \\ X_2 &= A_2 X_2 A_2^T + B_2 C_1 X_1 C_1^T B_2^T + \alpha_1 B_2 B_2^T. \end{aligned} \tag{4}$$

Proof: The system in Fig. 1 is mean-square stable iff for any uncorrelated white noise processes d_1 and d_2, which are uncorrelated with w_1 and w_2 and have variances $\alpha_1, \alpha_2 \in \mathbb{R}_+$, there exists a finite $\hat{X} \in M_+(\mathbb{R}^{n_{\hat{x}}})$ satisfying the following:

$$\begin{aligned} \hat{X} = &\begin{bmatrix} A_1 & \\ & A_2 \end{bmatrix} \hat{X} \begin{bmatrix} A_1 & \\ & A_2 \end{bmatrix}^T \\ &+ \begin{bmatrix} B_1 C_2 X_2 C_2^T B_1^T & \\ & B_2 C_1 X_1 C_1^T B_2^T \end{bmatrix} \\ &+ \begin{bmatrix} \alpha_2 B_1 B_1^T & \\ & \alpha_1 B_2 B_2^T \end{bmatrix}. \end{aligned} \tag{5}$$

Partition $\hat{X}$ according to the dimensions of x_1 and x_2 (the system states of $\mathcal{T}_1$ and $\mathcal{T}_2$, respectively) as

$$\hat{X} = \begin{bmatrix} X_1 & X_{21}^T \\ X_{21} & X_2 \end{bmatrix}.$$

The (2, 1) element of the above matrix obeying (5) satisfies the following:

$$X_{21} = A_2 X_{21} A_1^T.$$

Using Kronecker product, this equation is equivalent to

$$(I - A_1 \otimes A_2)\mathrm{vec}(X_{21}) = 0. \tag{6}$$

Since $\rho(A_1) < 1$ and $\rho(A_2) < 1$, hence

$$\rho(A_1 \otimes A_2) = \rho(A_1)\rho(A_2) < 1$$

which implies that $(I - A_1 \otimes A_2)$ is nonsingular, or the only solution for (6) is

$$\mathrm{vec}(X_{21}) = 0, \qquad \text{or } X_{21} = 0.$$

Therefore the mean-square stability of the system in Fig. 1 is equivalent to the existence of $X_1 \in M_+(\mathbb{R}^{n_{x_1}})$ and $X_2 \in M_+(\mathbb{R}^{n_{x_2}})$ satisfying (4). ∎

Definition 2.4: The interconnection in Fig. 1 is said to be internally mean-square stable if for any finite but uncorrelated white noise processes d_1 and d_2, which are uncorrelated with the stochastic multipliers of $\mathcal{T}_1$ and $\mathcal{T}_2$, the signals z_1 and z_2 in Fig. 2 belong to $\mathbf{v}$.

Lemma 2.5: Consider the interconnected system in Fig. 1, where $\mathcal{T}_1, \mathcal{T}_2 \in \boldsymbol{S}$. The following statements are equivalent:

iv) the system in Fig. 1 is mean-square stable;
v) the system in Fig. 2 is internally mean-square stable;
vi) $\|\mathcal{T}_1\|_\mathsf{s}\|\mathcal{T}_2\|_\mathsf{s} \neq 1$.

Proof: If i) holds then there exists X_1 and X_2 satisfying (4) by Theorem 2.3. Considering

$$\|z_i\|_\mathbf{v}^2 = C_i X_i C_i^T$$

therefore, $z_i \in \mathbf{v}$ for $i = 1, 2$, i.e., i) ⇒ ii). Now we want to show ii) ⇒ i). The following Lyapunov equation has a unique, finite and positive–definite solution L_i due to assumption A4)

$$L_i = A_i L_i A_i^T + B_i B_i^T$$

for $i = 1, 2$. Let x_1, x_2 be the system states of $\mathcal{T}_1$ and $\mathcal{T}_2$. The covariances of x_1 and x_2 for the systems $\mathcal{T}_1$ and $\mathcal{T}_2$ satisfy the following:

$$X_1 = \|d_2 + z_2\|_\mathbf{v}^2 L_1, \qquad X_2 = \|d_1 + z_1\|_\mathbf{v}^2 L_2. \tag{7}$$

Since $\|d_i + z_i\|_\mathbf{v}^2 \le \|d_i\|_\mathbf{v}^2 + \|z_i\|_\mathbf{v}^2$, hence $X_i \in M_+(\mathbb{R}^{n_{x_i}})$ and is finite if $\|z_i\|_\mathbf{v}$ is finite for $i = 1, 2$. Therefore ii)⇒ i). Now assume that ii) is true. From (3), it is not hard to see that $[d_1 \; d_2]^T$ and $\hat{x}$ are uncorrelated. Hence d_i is uncorrelated with z_i $(i = 1, 2)$, i.e., the following equations hold:

$$\begin{aligned} \|z_1\|_\mathbf{v}^2 &= \|\mathcal{T}_1\|_\mathsf{s}^2(\|d_2\|_\mathbf{v}^2 + \|z_2\|_\mathbf{v}^2) \\ \|z_2\|_\mathbf{v}^2 &= \|\mathcal{T}_2\|_\mathsf{s}^2(\|d_1\|_\mathbf{v}^2 + \|z_1\|_\mathbf{v}^2). \end{aligned} \tag{8}$$

A necessary and sufficient condition for the existence of finite and unique $\|z_1\|_\mathbf{v}$ and $\|z_2\|_\mathbf{v}$ to satisfy (8) is the norm condition iii). Therefore, ii) ⇔ iii). ∎

Define a unit ball $\boldsymbol{BS}$ in $\boldsymbol{S}$ as: $\boldsymbol{BS} \triangleq \{\Delta \in \boldsymbol{S}: \|\Delta\|_\mathsf{s} \le 1\}$. If $\mathcal{T}_1$ is any element in $\boldsymbol{BS}$, the following question is natural to ask: what condition should be posed on $\mathcal{T}_2$ such that the interconnection in Fig. 1 is mean square stable? That is, we want to find conditions such that a linear dynamic stochastic system will maintain MSS in the presence of norm-bounded dynamic stochastic uncertainties in $\boldsymbol{BS}$. The following result provides an answer to this question.

Theorem 2.6 (Mean-Square Small Gain Theorem): For any norm-bounded dynamic stochastic system $\mathcal{T}_1$ in $\boldsymbol{BS}$, the interconnection in Fig. 1 is mean-square stable iff $\|\mathcal{T}_2\|_\mathsf{s} < 1$.

Proof: $\|\mathcal{T}_2\|_\mathsf{s} < 1$ implies that iii) in lemma 2.5 is true, hence the sufficiency is obvious. Now we proceed to prove the necessity. Assume the interconnection in Fig. 1 is mean-square stable, but $\|\mathcal{T}_2\|_\mathsf{s} \ge 1$. Then there must exist a $\mathcal{T}_1 \in \boldsymbol{BS}$ such that $\|\mathcal{T}_1\|_\mathsf{s}\|\mathcal{T}_2\|_\mathsf{s} = 1$, one of such $\mathcal{T}_1$ can be constructed as $\mathcal{T}_1 = \mathcal{T}_2/\|\mathcal{T}_2\|_\mathsf{s}^2$. This implies that for this $\mathcal{T}_1$, there are no unique solutions for $\|z_1\|_\mathbf{v}$ and $\|z_2\|_\mathbf{v}$ in (8). This contradicts that the interconnection is mean-square stable. Hence, $\|\mathcal{T}_2\|_\mathsf{s} < 1$ is needed to guarantee that the interconnection in Fig. 1 is mean-square stable for all $\mathcal{T}_1 \in \boldsymbol{BS}$. ∎

Now we consider a multiloop feedback interconnection in Fig. 3. MSS and IMSS for the multiloop feedback interconnection can be similarly defined as in definition 2.2 and 2.4. Using i) and ii) in theorem 2.5 one loop at a time leads to the following result.

Corollary of Lemma 2.5: The multiloop feedback interconnection in Fig. 3 is mean-square stable iff it is internally mean-square stable.

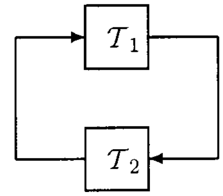

Fig. 1. The single loop feedback interconnection.

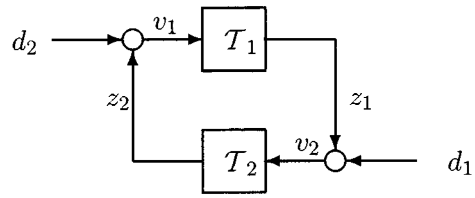

Fig. 2. Input–output signals in the single loop feedback interconnection.

Theorem 2.7 (Multiloop Mean-Square Small Gain Theorem): Suppose in Fig. 3, $\Delta_1, \Delta_2, \ldots, \Delta_n \in \boldsymbol{S}$, and $\mathcal{T}$ is a stochastic system with multiports

$$\begin{aligned} x_{k+1} &= Ax_k + \sum_{j=1}^{n}[B_j w_{j_k}]\, v_{j_k} \\ z_{i_k} &= C_j x_k + \sum_{j=1}^{n}[D_{ij} w_{j_k}]\, v_{j_k}, \qquad i \in \{1, 2, \ldots, n\} \\ v_{j_k} &= [\Delta_j(z_j)]_k, \qquad j \in \{1, 2, \ldots, n\} \end{aligned} \tag{9}$$

where $x \in \mathbb{R}^{n_x}$ is the system state, $z_j \in \mathbb{R}$ for $j \in \{1, \ldots, n\}$, and $w_1, w_2, \ldots, w_n$ are mutually uncorrelated white noise processes which are also uncorrelated with $\Delta_1, \Delta_2, \ldots, \Delta_n$. Denote T_{ij} as the transfer matrix of the system quadruple A, B_j, C_i, D_{ij} for $i, j \in \{1, 2, \ldots, n\}$. If

$$\rho[\mathtt{G}(\mathcal{T})\,\mathrm{diag}(\|\Delta_1\|_\mathsf{s}^2, \ldots, \|\Delta_n\|_\mathsf{s}^2)] < 1 \tag{10}$$

where

$$\mathtt{G}(\mathcal{T}) \triangleq \begin{bmatrix} \|T_{11}\|_2^2 & \cdots & \|T_{1n}\|_2^2 \\ \vdots & \ddots & \vdots \\ \|T_{n1}\|_2^2 & \cdots & \|T_{nn}\|_2^2 \end{bmatrix} \tag{11}$$

then the interconnection in Fig. 3 is internally mean-square stable.

Proof: Assume that the interconnection in Fig. 3 is internally mean-square stable, it must be mean-square stable. Let $X \in M_+(\mathbb{R}^{n_x})$ be such that

$$X = X_0 + \sum_{j=1}^{n}\sum_{k=0}^{\infty} A^k B_j B_j^T A^{T^k} \|\Delta_j\|_\mathsf{s}^2 \|z_j\|_\mathbf{v}^2$$

where

$$X_0 = \sum_{k=0}^{\infty}\sum_{j=1}^{n} A^k B_j B_j^T A^{T^k} \|d_j\|_\mathbf{v}^2.$$

In this case, $\|z_i\|_\mathbf{v}$ satisfies

$$\begin{aligned} \|z_i\|_\mathbf{v}^2 = {} & C_i X_0 C_i^T + \sum_{j=1}^{n} D_{ij}^2 \|d_j\|_\mathbf{v} \\ & + \sum_{j=1}^{n}\left(\sum_{k=0}^{\infty} C_i A^k B_j B_j^T A^{T^k} C_i^T + D_{ij}^2\right)\|\Delta_i\|_\mathsf{s}^2\|z_i\|_\mathbf{v}^2 \end{aligned}$$

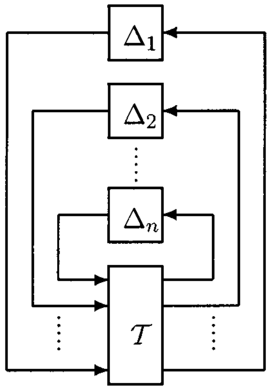

Fig. 3. The multiloop feedback interconnection.

for $i \in \{1, 2, \ldots, n\}$ or in matrix form

$$\begin{bmatrix} \|z_1\|_{\mathtt{v}}^2 \\ \|z_2\|_{\mathtt{v}}^2 \\ \vdots \\ \|z_n\|_{\mathtt{v}}^2 \end{bmatrix} = \mathtt{G}(\mathcal{T}) \begin{bmatrix} \|\Delta_1\|_{\mathtt{s}}^2 \|z_1\|_{\mathtt{v}}^2 \\ \|\Delta_2\|_{\mathtt{s}}^2 \|z_2\|_{\mathtt{v}}^2 \\ \vdots \\ \|\Delta_n\|_{\mathtt{s}}^2 \|z_n\|_{\mathtt{v}}^2 \end{bmatrix} + \mathtt{G}(\mathcal{T}) \begin{bmatrix} \|d_1\|_{\mathtt{v}}^2 \\ \|d_2\|_{\mathtt{v}}^2 \\ \vdots \\ \|d_n\|_{\mathtt{v}}^2 \end{bmatrix}. \tag{12}$$

There is a unique and finite solution for $\|z_1\|_{\mathtt{v}}, \|z_2\|_{\mathtt{v}}, \ldots, \|z_n\|_{\mathtt{v}}$ in (12) iff

$$I - \mathtt{G}(\mathcal{T}) \operatorname{diag}(\|\Delta_1\|_{\mathtt{s}}^2, \ldots, \|\Delta_n\|_{\mathtt{s}}^2)$$

is invertible in $M(\mathbb{R}_+^n)$. A sufficient condition to guarantee this is (10). ■

III. Robust Control System Analysis

Let us add an external disturbance channel and an output channel to (9) and call the resultant system $\hat{\mathcal{T}}$

$$\begin{aligned} x_{k+1} &= Ax_k + B_0 v_{0_k} + \sum_{j=1}^n [B_j w_{j_k}] v_{j_k} \\ z_{0_k} &= C_0 x_k + D_{j0} v_{0_k} + \sum_{j=1}^n [D_{0j} w_{j_k}] v_{j_k} \\ z_{i_k} &= C_i x_k + D_{i0} v_{0_k} + \sum_{j=1}^n [D_{ij} w_{j_k}] v_{j_k} \\ & i \in \{1, 2, \ldots, n\} \\ v_{j_k} &= [\Delta_j(z_j)]_k \end{aligned} \tag{13}$$

where $w_1, w_2, \ldots, w_n$ are mutually uncorrelated white noise processes and $\Delta_i \in \boldsymbol{BS}$ for $i \in \{1, \ldots, n\}$ with system input z_i; v_0 is a white noise process with unit covariance. Denote

$$\boldsymbol{\Delta} \triangleq \{\operatorname{diag}(\Delta_1, \Delta_2, \ldots, \Delta_n) \colon \Delta_i \in \boldsymbol{BS}, \qquad i \in \{1, \ldots, n\}.$$

Definition 3.1: The system (13) is robustly stable in the mean square sense with respect to $\boldsymbol{\Delta}$ if for any given norm-bounded dynamic stochastic perturbations $\Delta_i \in \boldsymbol{BS}$, $i \in \{1, \ldots, n\}$, the multiloop interconnection in Fig. 3 is internally mean-square stable.

Theorem 3.2: The system (13) is robustly stable in mean square sense with respect to the norm-bounded dynamic stochastic perturbations in $\boldsymbol{BS}$ iff the following holds:

$$\phi(\mathcal{T}, \boldsymbol{\Delta}) \triangleq \rho(\mathtt{G}(\mathcal{T})) < 1$$

where $\mathtt{G}(\mathcal{T})$ is defined in (11) and $\mathcal{T}$ is the stochastic system by dropping the disturbance channel and the output channel z_0, i.e., the system depicted in (9).

Proof: From mean-square small gain theorem, we know that

$$\rho[\mathtt{G}(\mathcal{T}) \operatorname{diag}(\|\Delta_1\|_{\mathtt{s}}^2, \ldots, \|\Delta_n\|_{\mathtt{s}}^2)] < 1 \qquad \forall\, \Delta_i \in \boldsymbol{BS},\ i \in \{1, \ldots, n\} \tag{14}$$

is sufficient for (13) to be robustly stable in mean square sense.

Denote

$$\phi(\mathcal{T}, \boldsymbol{\Delta}) = \max_{\Delta_i \in \boldsymbol{BS},\, 1 \le i \le n} \rho[\mathtt{G}(\mathcal{T}) \operatorname{diag}(\|\Delta_1\|_{\mathtt{s}}^2, \ldots, \|\Delta_n\|_{\mathtt{s}}^2)] \tag{15}$$

then (14) is equivalent to

$$\phi(\mathcal{T}, \boldsymbol{\Delta}) < 1. \tag{16}$$

Now, we need to prove that (16) is also a necessary condition. Consider that

$$\mathtt{G}(\mathcal{T}) \operatorname{diag}(\|\Delta_1\|_{\mathtt{s}}^2, \ldots, \|\Delta_n\|_{\mathtt{s}}^2)$$

is a matrix of all positive elements, hence its spectral radius is its largest eigenvalue [2, Ch. 8]. Hence, (16) is equivalent to

$$\max_{\Delta_i \in \boldsymbol{BS},\, 1 \le i \le n} \overline{\lambda}[\mathtt{G}(\mathcal{T}) \operatorname{diag}(\|\Delta_1\|_{\mathtt{s}}^2, \ldots, \|\Delta_n\|_{\mathtt{s}}^2)] < 1.$$

Assume that (16) fails, i.e., there exists a set of $\Delta_i \in \boldsymbol{BS}$, $i \in \{1, \ldots, n\}$ such that

$$\overline{\lambda}[\mathtt{G}(\mathcal{T}) \operatorname{diag}(\|\Delta_1\|_{\mathtt{s}}^2, \ldots, \|\Delta_n\|_{\mathtt{s}}^2] \not< 1.$$

Since $0 \in \boldsymbol{BS}$ and $\overline{\lambda}(\cdot)$ is a continuous function of its arguments, there must exist another set of $\hat{\Delta}_i \in \boldsymbol{BS}$, $i \in \{1, \ldots, n\}$ such that

$$\overline{\lambda}[\mathtt{G}(\mathcal{T}) \operatorname{diag}(\|\hat{\Delta}_1\|_{\mathtt{s}}^2, \ldots, \|\hat{\Delta}_n\|_{\mathtt{s}}^2)] = 1$$

or

$$I - \mathtt{G}(\mathcal{T}) \operatorname{diag}(\|\hat{\Delta}_1\|_{\mathtt{s}}^2, \ldots, \|\hat{\Delta}_n\|_{\mathtt{s}}^2)$$

is singular. This implies that (12) will not have a unique solution $\|z_i\|_{\mathtt{v}}$s with respect to this set of $\hat{\Delta}_i \in \boldsymbol{BS}$, $i \in \{1, \ldots, n\}$. Therefore if (16) fails, (13) is not internally mean-square stable for some $\hat{\Delta}_i \in \boldsymbol{BS}$, $i \in \{1, \ldots, n\}$. Hence, (13) is robustly stable in mean-square sense iff (16) holds. In order to complete the proof, we need to show that the quantity defined in (15) satisfies

$$\phi(\mathcal{T}, \boldsymbol{\Delta}) = \rho(\mathtt{G}(\mathcal{T})).$$

Denote the ith row of $\mathtt{G}(\mathcal{T})$ as $[\mathtt{G}]_{i*}$. Using [2, Cor. 8.1.31], we have

$$\begin{aligned} \phi(\mathcal{T}, \boldsymbol{\Delta}) &= \max_{\Delta_i \in \boldsymbol{BS},\, 1 \le i \le n} \max_{e \in \mathbb{R}_+^n} \min_{1 \le i \le n,\, e_i \ne 0} \\ &\quad \cdot \left\{ [\mathtt{G}]_{i*} \operatorname{diag}(\|\Delta_1\|_{\mathtt{s}}^2, \ldots, \|\Delta_n\|_{\mathtt{s}}^2) \frac{e}{e_i} \right\} \\ &= \max_{e \in \mathbb{R}_+^n} \max_{\Delta_i \in \boldsymbol{BS},\, 1 \le i \le n} \min_{1 \le i \le n,\, e_i \ne 0} \\ &\quad \cdot \left\{ [\mathtt{G}]_{i*} \operatorname{diag}(\|\Delta_1\|_{\mathtt{s}}^2, \ldots, \|\Delta_n\|_{\mathtt{s}}^2) \frac{e}{e_i} \right\} \\ &\le \max_{e \in \mathbb{R}_+^n} \min_{1 \le i \le n,\, e_i \ne 0} \max_{\Delta_i \in \boldsymbol{BS},\, 1 \le i \le n} \\ &\quad \cdot \left\{ [\mathtt{G}]_{i*} \operatorname{diag}(\|\Delta_1\|_{\mathtt{s}}^2, \ldots, \|\Delta_n\|_{\mathtt{s}}^2) \frac{e}{e_i} \right\} \\ &= \max_{e \in \mathbb{R}_+^n} \min_{1 \le i \le n,\, e_i \ne 0} \\ &\quad \cdot \left\{ [\mathtt{G}]_{i*} \frac{e}{e_i} \right\} \\ &= \overline{\lambda}(\mathtt{G}(\mathcal{T})). \end{aligned}$$

Notice that the third inequality comes from the following standard result for multivariable functions:

$$\max_y \min_x f(x, y) \le \min_x \max_y f(x, y).$$

The fourth equality comes from the maximization of a linear function with positive arguments. On the other hand, $\Delta_i \in \boldsymbol{BS}, i \in \{1, \ldots, n\}$ implies

$$\rho(\mathtt{G}(\mathcal{T})) \le \phi(\mathcal{T}, \boldsymbol{\Delta}).$$

Hence, we must have

$$\phi(\mathcal{T}, \boldsymbol{\Delta}) = \rho(\mathtt{G}(\mathcal{T})) = \overline{\lambda}(\mathtt{G}(\mathcal{T})).$$

■

Definition 3.3: The system (13) is said to have robust performance with respect to $\boldsymbol{BS}$ if it is robustly stable in mean square sense and for all norm-bounded dynamic stochastic perturbations $\Delta_i \in \boldsymbol{BS}, i \in \{1, \ldots, n\}$, the output variance of z_0 in (13) is bounded

$$\mathbf{E}_\infty[\|z_0\|^2] = \|z_0\|_{\mathtt{v}}^2 < 1.$$

Theorem 3.4: The system (13) has robust performance with respect to $\boldsymbol{BS}$ iff

$$\phi(\hat{\mathcal{T}}, \hat{\boldsymbol{\Delta}}) \triangleq \rho(\mathtt{G}(\hat{\mathcal{T}})) < 1$$

where

$$\mathtt{G}(\hat{\mathcal{T}}) \triangleq \begin{bmatrix} \|T_{00}\|_2^2 & \cdots & \|T_{0n}\|_2^2 \\ \vdots & \ddots & \vdots \\ \|T_{n0}\|_2^2 & \cdots & \|T_{nn}\|_2^2 \end{bmatrix} \tag{17}$$

and

$$\hat{\boldsymbol{\Delta}} \triangleq \{\mathrm{diag}(\Delta_0, \Delta_1, \ldots, \Delta_n), \Delta_i \in \boldsymbol{BS}, i \in \{0, \ldots, n\}\}.$$

Proof: The robust stability of (13) in mean-square sense implies the following equality holds $\forall \Delta_i \in \boldsymbol{BS}, i \in \{1, \ldots, n\}$:

$$\begin{bmatrix} \|z_1\|_{\mathtt{v}}^2 \\ \|z_2\|_{\mathtt{v}}^2 \\ \vdots \\ \|z_n\|_{\mathtt{v}}^2 \end{bmatrix} = \mathtt{G}(\mathcal{T}) \begin{bmatrix} \|\Delta_1\|_{\mathtt{s}}^2 \|z_1\|_{\mathtt{v}}^2 \\ \|\Delta_2\|_{\mathtt{s}}^2 \|z_2\|_{\mathtt{v}}^2 \\ \vdots \\ \|\Delta_n\|_{\mathtt{s}}^2 \|z_n\|_{\mathtt{v}}^2 \end{bmatrix} + \begin{bmatrix} \|T_{10}\|_2^2 \\ \|T_{20}\|_2^2 \\ \vdots \\ \|T_{n0}\|_2^2 \end{bmatrix} \tag{18}$$

and the output variance of z_0 can be computed as

$$\|z_0\|_{\mathtt{v}}^2 = \|T_{00}\|_{\mathtt{s}}^2 + [\, \|T_{01}\|_{\mathtt{s}}^2 \quad \|T_{02}\|_{\mathtt{s}}^2 \quad \cdots \quad \|T_{0n}\|_{\mathtt{s}}^2 \,] \cdot \begin{bmatrix} \|\Delta_1\|_{\mathtt{s}}^2 \|z_1\|_{\mathtt{v}}^2 \\ \|\Delta_2\|_{\mathtt{s}}^2 \|z_2\|_{\mathtt{v}}^2 \\ \vdots \\ \|\Delta_n\|_{\mathtt{s}}^2 \|z_n\|_{\mathtt{v}}^2 \end{bmatrix}. \tag{19}$$

The robust MSS is equivalent to $\forall \Delta_i \in \boldsymbol{BS}, i \in \{1, \ldots, n\}$

$$\det[I - \mathtt{G}(\mathcal{T})\, \mathrm{diag}(\|\Delta_1\|_{\mathtt{s}}^2, \ldots, \|\Delta_n\|_{\mathtt{s}}^2)] > 0$$

and performance bounding is equivalent to

$$1 - \|z_0\|_{\mathtt{v}}^2 > 0, \qquad \forall \Delta_i \in \boldsymbol{BS}, i \in \{1, \ldots, n\}.$$

The combination of those two is equivalent to $\forall \Delta_i \in \boldsymbol{BS}, i \in \{0, \ldots, n\}$

$$\det[I - \mathtt{G}(\hat{\mathcal{T}})\, \mathrm{diag}(\|\Delta_0\|_{\mathtt{s}}^2, \|\Delta_1\|_{\mathtt{s}}^2, \ldots, \|\Delta_n\|_{\mathtt{s}}^2)] > 0$$

which is further equivalent to $\phi(\hat{\mathcal{T}}, \hat{\boldsymbol{\Delta}}) < 1$. Hence, the claim follows. ■

IV. Conclusion

The mean-square small gain theorem characterizes conditions for the mean-square stability of a class of stochastic systems. This characterization provides a method for robust control analysis for systems with dynamic stochastic uncertainties. Both robust stability and performance conditions can be characterized by the proposed small gain theorem. This characterization will facilitate the output feedback control law synthesis. Further studies are needed to extend the mean-square small-gain theorem to more general systems.

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