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Iterative Identification and Control Design Using Finite-signal-to-noise Models *

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ABSTRACT

In this paper, a model is said to be *validated for control design* if using the model-based controller, the closed loop performance of the real plant satisfies a specified performance bound. To improve the model for control design, only closed loop response data is available to deduce a new model of the plant. Hence the procedure described herein involves three steps in each iteration: (i) closed loop identification; (ii) plant model extraction from the closed loop model; (iii) controller design. Thus our criteria for model validation involve both the control design procedure by which the closed loop system performance is evaluated, and the identification procedure by which a new model of the plant is deduced from the closed loop response data. This paper proposes new methods for both parts, and also proposes an iterative algorithm to connect the two parts. To facilitate both the identification and control tasks, the new finite-signal-to-noise (FSN) model of linear systems is utilized. The FSN model allows errors in variables whose noise covariances are proportional to signal covariances. Allowing the signal to noise ratios to be bounded but uncertain, a control theory to guarantee a variance upper bound is developed for the discrete version of this new FSN model. The identification of the closed loop system is accomplished by a new type of q-Markov Cover, adjusted to accommodate the assumed FSN structure of the model. The model of the plant is extracted from the closed loop identification model. This model is then used for control design and the process is repeated until the closed loop performance validates the model. If the iterations produce no such a controller, we say that this specific procedure cannot produce a model *valid for control design* and the level of the required performance must be reduced.

Key words: robust control, system identification.

NOTATION

In this paper, $\mathbb{R}$ and $\mathbb{R}_+$ denote the sets of real and positive real numbers. diag$(\cdot)$ is the diagonal matrix whose elements come from the vector $(\cdot)$. ddiag$(\cdot)$ is the diagonal matrix whose elements come from the diagonal of the matrix $(\cdot)$ (using Matlab commands, ddiag$(\cdot)$ is the same as diag(diag$(\cdot)$)). $\{(\cdot)_{ij}\}$

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represents a matrix whose (i,j)-element is $(\cdot)_{ij}$, E_i is a square matrix with 1 at its (i,i)-location, and 0 at all the other locations. $\otimes$ denotes the standard Kronecker product. $\bar{\lambda}(\cdot)$ is the largest eigenvalue of a matrix $(\cdot)$. $\mathcal{E}_\infty$ is the steady state expectation operator.

The script capital letters are used to represent the dynamical systems, $\mathcal{P}$ and $\mathcal{K}$ denote the plant and controller dynamical systems. The closed loop system, composed of the plant and control pair $(\mathcal{P},\mathcal{K})$ in Fig. 2 is denoted as $\mathcal{T}(\mathcal{P},\mathcal{K})$ and

$$\text{relates } \begin{bmatrix} v_c \\ v_z \end{bmatrix} \text{ to } \begin{bmatrix} u \\ z \end{bmatrix}.$$

If we want to add more output variables in $\mathcal{T}(\mathcal{P},\mathcal{K})$, we denote this closed loop system as $\tilde{\mathcal{T}}(\mathcal{P},\mathcal{K})$.

The identified models are represented by using superscript $\hat{(\cdot)}$. For example, $\hat{\mathcal{P}}$ and $\hat{\mathcal{T}}(\mathcal{P},\mathcal{K})$ are the identified models of the plant $\mathcal{P}$ and the closed loop system $\mathcal{T}(\mathcal{P},\mathcal{K})$.

Associated with any given linear dynamical system $\mathcal{P}$ having the transfer function matrix

$$D + C(sI - A)^{-1}B,$$

the system matrix of $\mathcal{P}$ is defined as

$$\begin{bmatrix} D & C \\ B & A \end{bmatrix}.$$

For given stochastic signals v and y, the autocorrelations of the signal y are defined as

$$R_i \triangleq \mathcal{E}_\infty\, y(k+i)y(k)^T, \;\; i = 0, 1, \cdots.$$

The crosscorrelations of the signal y and v are defined as

$$R_{iyv} \triangleq \mathcal{E}_\infty\, y(k+i)v(k)^T, \;\; i = 0, 1, \cdots.$$

1 INTRODUCTION

The vast majority of industrial controllers are PID and are not model-based. While this avoids reliance on models and model error characterizations, these controllers may not yield performance adequate for high precision control.

Let $\mathcal{P}$ denote a real dynamical system, hence it is not subject to exact mathematical description. Assume $\hat{\mathcal{P}}$ is an approximation of $\mathcal{P}$ (for instance, from an identification algorithm). The discrepancy between $\mathcal{P}$ and $\hat{\mathcal{P}}$ is usually denoted as $\Delta\mathcal{P} = \mathcal{P} - \hat{\mathcal{P}}$. $\Delta\mathcal{P}$ cannot be completely known. Instead, the norm bound γ of $\Delta\mathcal{P}$ is usually estimated. Robust control theory can find a controller such that the whole set of the plants described by

$$\{\hat{\mathcal{P}} + \Delta\mathcal{P} \ : \ \|\Delta\mathcal{P}\| < \gamma\} \tag{1}$$

are stabilized. The robustness of such a controller rests on the fact that it controls infinitely many plants in the set (1). The size of this set depends on the size of γ. The larger the γ, the more robust is the controller. Normally performance and robustness are tradeoffs (performance goes down as γ goes up). Determining that

$$\hat{\mathcal{P}} + \Delta\mathcal{P}$$

is consistent with observed data for some γ such that

$$\|\Delta\mathcal{P}\| < \gamma$$

is often called "model validation". However, the usefulness of robust control and model validation concepts relate to the following questions:

- Does an infinite error $\Delta\mathcal{P}$ in the plant model imply that the plant model is bad for control design?
- Does arbitrarily small $\Delta\mathcal{P}$ imply that the plant model is good for control design?

Reference [3] gives examples to show that the answer to both questions is no, due to the fact that the modeling and control problems are not independent. Hence, a small $\Delta\mathcal{P}$ does not necessarily lead to good closed loop performance, and a large $\Delta\mathcal{P}$ (and hence large γ) does not necessarily lead to bad closed loop performance. Hence robust control theory only solves control problems for a small fraction of the desirable plant descriptions, namely the plant descriptions with small open loop plant perturbations. Other techniques might not compromise performance even for large γ. Even robust identification coupled with robust control may not yield an appropriate controller due to the conservative nature of the results, and due to the fact that no closed loop identification is used to improve the models. Only closed loop criteria for model validation can remedy these situations.

Alternative approaches, called *iterative identification and control* have been proposed. [4, 13] are two examples of such approaches. The control design is not a robust control design, although some robustness is indirectively achieved because of the iterative nature for model improvement. [20] stabilizes both the reduced order model and the real plant. Our motivation is to iteratively combine identification with robust control. Similar motivation has led [21] to use frequency domain approaches. Our work is in time domain.

The uncertainty considered in our paper is characterized in the so-called *finite-signal-to-noise* (FSN) model [1], where sensor and actuator noises are modeled as zero-mean, white noises with variances affinely related to the signal variances. The ratio of the signal variance to the noise variance is defined as the "signal to noise ratio" (SNR). The SNR is unknown but bounded in our robust control problem (called herein,"robust FSN control"). This paper introduces the theory of robust FSN control. It includes some multiplicative noises as special cases, and hence can represent some parameter uncertainty. Hence our approach here also solves certain deterministic robust stabilization problem [22].

Some comment is required on the definitions of model "validation" and "invalidation". If one means by "validation" that the model $\hat{\mathcal{P}} + \Delta\mathcal{P}$, $\|\Delta\mathcal{P}\| < \gamma$ is consistent with all experimental data that one might conceive, then any mathematical model can always be "invalidated" by an experiment [3], since mathematical models can never capture all input/output properties of a real system. The model validation in [18, 16, 17] uses this meaning for robust control design. Their approach characterizes model error in the open loop sense and focuses mainly on how to evaluate the consistency between the perturbed plant and the observed data. No intention is given to improving the model when the inconsistency does happen. Our intent and our meaning of the word "validation" is that a specified closed loop performance is achieved using the model (and possibly an assumed error bound) in a model based controller. To distinguish from other definitions we will call this "performance validation" of models. The integrated open-loop identification and control studied in [5] provides a method to do "performance validation" of models by using weighted q-Markov Cover identification and Output Covariance Control (OCC). The weight obtained in the OCC control design is used as the weight for the open loop identification and this weight passes important performance-relevant information to identification, such that a proper model can be found for control design. The integrated closed loop identification and control is another "performance validation" of models where the model error is considered in the closed loop sense, and the closed loop performance depends on both model refinement and control redesign. Such approaches [3, 5, 8, 9, 10, 11, 12, 13, 14] seek to iteratively improve performance, unlike the robust control scenario where model is not updated.

This paper provides a method to do performance validation of models and control design, complementary to the control method in [2]. We update the control design model $\mathcal{P}$ based upon closed loop identification. A new controller is designed on each new iteration. For our example the process converges in five iterations. The convergence of any iterative approach depends upon both the choice of identification and control methods. Rapidly achieving performance specifications along iterations would indicate either lenient performance criteria or coherence between the identification and control methods. This coherence is an important new research objective and is the motivation of this paper. To address this coherence issue, we modify both identification and control methods to make them more compatible. The new identification results extend the q-Markov Cover method to include measurement and actuation noise (errors in variables). Furthermore, this noise may have a finite signal-to-noise (FSN) structure, where larger signals carry more noise. The new control design result introduced in [2] and modified here for the discrete case, utilizes the FSN structure to parameterize model error as the signal-to-noise ratios in all input/output channels. This parameterization of model error makes the identification and control steps more compatible, leading to better results in fewer iterations. In addition, the FSN noise model is much more realistic in practical situations. As examples, the roundoff error in A/D conversion is related to the size of the signal, and the turbulence noise on an aircraft is related to the angle of attack signal.

This paper is organized as the follows. Section 2 describes the new FSN model structure for a real controlled system. Section 3 describes the new identification method which is called $\mathrm{QMC}_{\mathrm{fsn}}$. This identification will be used in the integrated process, and performed for closed loop systems. Section 4 describes the method to extract the plant model from closed loop data. Section 5 describes the control design algorithm using the new control design method. This control design algorithm provides an output feedback controller for a linear FSN system. Section 6 describes the iterative algorithm which combines closed loop identification and control design. Section 7 describes an example. Section 8 offers some conclusions.

2 FINITE SIGNAL-TO-NOISE MODEL STRUCTURE

A controlled system consists of a plant $\mathcal{P}$, a control computer $\mathcal{K}$, actuating hardware $\mathcal{A}$, sensing hardware $\mathcal{S}$, and A/D and D/A converters. Noises enter each of these components. In the following Fig. 1, the block diagram depicts a typical controlled system. Where $v_c \in \mathbb{R}^m$ and $v_z \in \mathbb{R}^n$ are signals injected by the control computer (say for identification purposes); $u \in \mathbb{R}^m$ and $z \in \mathbb{R}^n$ are signals which can be measured through the control computer; w_p, w_a and w_s are noises associated with the original plant $\mathcal{P}$, the actuating hardware $\mathcal{A}$ and sensing hardware $\mathcal{S}$; w_z represents the quantization error in A/D conversion and w_u the quantization error in D/A conversion; w_c is the controller state computational error. In this paper we only consider noises at sources v_c, v_z, w_u, w_z and w_c. The classical model for these noises is the additive white noise, which is independent of system signals. However, FSN models described below are more realistic [1].

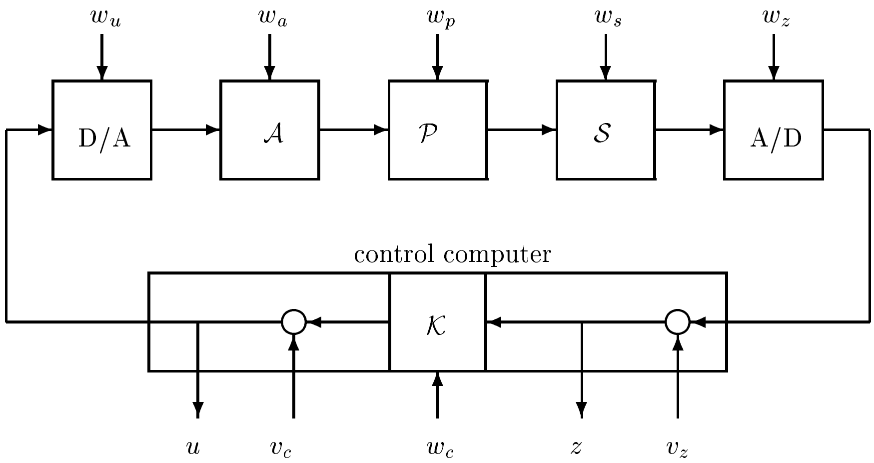

Fig. 1: A Controlled System with Noises

In Fig. 1, let the noise w_u be zero-mean white noise with covariance W_u and $u(k)$ be the control signal with covariance U. In the majority of linear systems (including LQG control theory), W_u is unrelated to any measure of strength of the signal u. However, the FSN model assumes that they are affinely related:

$$W_{u_{ii}} = W_{uo_{ii}} + \delta_i U_{ii}. \tag{2}$$

That is, the signal-to-noise ratio of the i-th control channel defined by

$$SNR \triangleq \delta_i^{-1} \triangleq \frac{U_{ii}}{(W_u - W_{uo})_{ii}}$$

is finite in the FSN model and infinite in the classical model. It is easy to understand why feedback controllers using FSN models yield maximal accuracy at finite control gains. Conversely, the LQG theory uses a noise model that leads to maximal accuracy at infinite control gains. This is one unfortunate and unrealistic property of LQG controllers. A minimum variance FSN controller is one that yields minimum output covariance (or variance) using FSN noise models, characterized by property (2).

Ignoring w_p and any dynamics in D/A, $\mathcal{A}$, $\mathcal{S}$ and A/D and moving the noise source w_c to the input and output of the control computer leads to the following block diagram

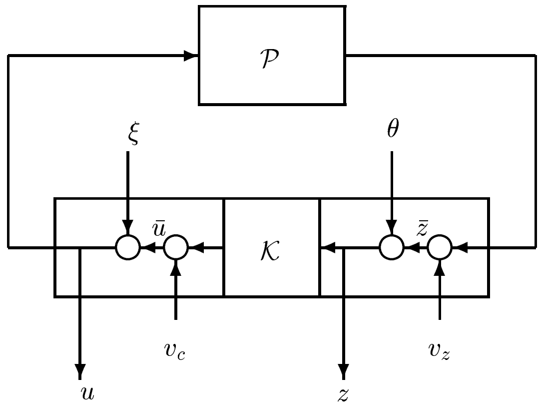

Fig. 2: Setup for Closed Loop Identification

where $\xi \in \mathbb{R}^m$, $\theta \in \mathbb{R}^n$ are equivalent or approximant descriptions of noises w_u, w_z and w_c. Define $\bar{z}$ and $\bar{u}$ by $z = \bar{z} + \theta$ and $u = \bar{u} + \xi$ as the input signals and let the covariance of a signal $x(t)$ be denoted by its capital X. Hence

$$U = \bar{U} + \Xi, \quad Z = \bar{Z} + \Theta \tag{3}$$

where the plant $\mathcal{P}$ and the control computer are characterized as discrete-time systems. Hence, all the noises ξ, θ, v_c and v_z are independent discrete-time *white noise processes*. The discrete-time white noise process is assumed to be a wide sense stationary process. This paper assumes FSN noise models for $[\xi^T \;\; \theta^T]^T$.

The following assumptions will be used:

Assumption (A1)∼(A3):

(A1) *The identification signals v_z and v_c are discrete-time white noise processes with known covariances.*

(A2) *ξ and θ are discrete-time FSN white noise processes, i.e., if we denote the covariance of ξ, θ as Ξ, Θ and the covariance of $\bar{u}$, $\bar{z}$ as $\bar{U}$ and $\bar{Z}$, then the FSN assumption implies*

$$\Xi = \Xi_0 + \mathrm{diag}(\delta_c)\,\mathrm{ddiag}(\bar{U}) \tag{4}$$
$$\Theta = \Theta_0 + \mathrm{diag}(\delta_z)\,\mathrm{ddiag}(\bar{Z}) \tag{5}$$

where

$$\delta_c = [\delta_{c_1}, \cdots, \delta_{c_{n_c}}]^T$$
$$\delta_z = [\delta_{z_1}, \cdots, \delta_{z_{n_z}}]^T$$

and δ_{c_i} and δ_{z_j} represent the noise-to-signal ratios at each channel. We assume that the noise-to-signal ratios are all less than 1 [1]*, i.e., we assume*

$$\begin{aligned} 0 \le \delta_{c_i} \le \delta_{c_i}^{+} < 1, \;\; i = 1, \cdots, m \\ 0 \le \delta_{z_j} \le \delta_{z_j}^{+} < 1, \;\; j = 1, \cdots, n \end{aligned} \tag{6}$$

Ξ_0 and Θ_0 are known constant matrices which characterize the part of the covariance (of the ambient noise) that is not proportional to the signal covariance. Notice that $\bar{U}$ and $\bar{Z}$ satisfy (3), hence Ξ and Θ in (5) satisfy

$$\begin{aligned} \Xi + \mathrm{diag}(\delta_c)\,\mathrm{ddiag}(\Xi) = \Xi_0 + \mathrm{diag}(\delta_c)\,\mathrm{ddiag}(U) \\ \Theta + \mathrm{diag}(\delta_z)\,\mathrm{ddiag}(\Theta) = \Theta_0 + \mathrm{diag}(\delta_z)\,\mathrm{ddiag}(Z) \end{aligned}$$

(A3) *All the processes ξ, θ, v_c and v_z are mutually independent.*

3 CLOSED-LOOP IDENTIFICATION FROM NOISY DATA

Now we consider the real closed loop system in Fig. 3 which has FSN noise and can be depicted by the following block diagram

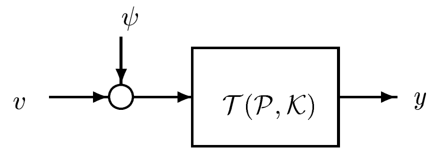

Fig. 3: The block diagram for identification purposes.

where $\mathcal{T}(\mathcal{P}, \mathcal{K})$ denotes the system relating $v + \psi$ to y and

$$v = \begin{bmatrix} v_c \\ v_z \end{bmatrix}, \;\; \psi = \begin{bmatrix} \xi \\ \theta \end{bmatrix}, \;\; y = \begin{bmatrix} u \\ z \end{bmatrix}.$$

We wish to construct a state space model of the closed loop system using only an identification experiment with white noise inputs. Following that, the plant model must be extracted from this closed loop model. This second task is treated in section 4.

[1]This implies that the noise level is at most the same as the level of the signal. This is not a necessary assumption.

The identification method used here finds a linear state space model to match the data set

$$\text{Data}_q \triangleq \{R_i, R_{iyv},\ i = 0, 1, \cdots, q-1\}$$

where q is a chosen integer, R_i and R_{iyv} are computed from the noise response of the real system

$$\begin{aligned} R_i &\triangleq \mathcal{E}_\infty\, y(k+i)y(k)^T \\ R_{iyv} &\triangleq \mathcal{E}_\infty\, y(k+i)v(k)^T, \quad i = 0, 1, \cdots, q-1. \end{aligned}$$

A linear model (A, B, C, D) which matches the data set Data_q is called q-Markov Cover. When no measurement noise is present, necessary and sufficient conditions for the existence of q-Markov Covers are given in [6, 7] as well as a parameterization of all q-Markov Covers. A new contribution of this section is to provide a parameterization of all q-Markov Covers for systems with input and output noises of FSN structure. We will call such models QMC_{fsn}.

Construct Toeplitz matrices

$$\begin{aligned} \mathsf{R}_q &\triangleq \begin{bmatrix} R_0 & R_1^T & \cdots & R_{q-1}^T \\ R_1 & R_0 & \cdots & R_{q-2}^T \\ \vdots & \vdots & \ddots & \vdots \\ R_{q-1} & R_{q-2} & \cdots & R_0 \end{bmatrix} \\ \mathsf{R}_{qyv} &\triangleq \begin{bmatrix} R_{0yv} & 0 & \cdots & 0 \\ R_{1yv} & R_{0yv} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ R_{(q-1)yv} & R_{(q-2)yv} & \cdots & R_{0yv} \end{bmatrix}. \end{aligned}$$

Also denote the covariance of $v(k)$ as V. Define

$$\begin{aligned} \Psi &= \begin{bmatrix} \Xi & 0 \\ 0 & \Theta \end{bmatrix} \\ \Psi_0 &= \begin{bmatrix} \Xi_0 & 0 \\ 0 & \Theta_0 \end{bmatrix} \end{aligned}$$

and

$$\delta = [\delta_c^T \ \ \delta_z^T]^T.$$

From assumption (A2), the FSN noise $\psi(k)$ which is generated in the closed loop system (Fig. 2) has the covariance satisfying

$$\Psi + \text{diag}(\delta)\,\text{ddiag}(\Psi) = \Psi_0 + \text{diag}(\delta)\,\text{ddiag}(R_0) \tag{7}$$

The following theorem determines when a $\mathrm{QMC_{fsn}}$ exists.

Theorem 3.1: *Suppose assumptions (A1)∼(A3) hold, and the data set* Data_q *is given. Then the following statements are equivalent:*

(i) *There exists a model of FSN structure which matches the noisy data set* Data_q.

(ii) *The following holds*

$$\mathsf{R}_q \geq \mathsf{R}_{qyv}\mathsf{V}^{-1}(\mathsf{V} + I \otimes \Psi)\mathsf{V}^{-T}\mathsf{R}_{qyv}^T$$

where $\mathsf{V} = I \otimes V$ *and* Ψ *satisfies (7).*

Proof: See appendix A.

Note that the existence of a q-Markov Cover depends on the choice of q. The largest q for which a q-Markov Cover exists is easily computed by increasing q until condition (ii) fails. For large q see [5] to construct an approximate q-Markov Cover which is closest to the data.

For the system in Fig. 3, consider the following algorithm

The $\mathrm{QMC_{fsn}}$ Algorithm (q-Markov Cover of FSN Structure):

Step 1 *Compute the data set* Data_q, U, *and* V *from the noisy data* $v(k)$ *and* $y(k)$ $(k = 0, 1, \cdots, l)$.

Step 2 *Compute* Ψ *from the equality (7) and compute the data matrix*

$$\mathsf{D}_q = \mathsf{R}_q - \mathsf{R}_{qyv}\mathsf{V}^{-1}(\mathsf{V} + I \otimes \Psi)\mathsf{V}^{-T}\mathsf{R}_{qyv}^T$$

if $\mathsf{D}_q \geq 0$, *find a full rank matrix factor* O_q: $\mathsf{D}_q = \mathsf{O}_q\mathsf{O}_q^T$. *If* $\mathsf{D}_q \not\geq 0$, *there is no linear model matching the data set* Data_q *and stop.*

Step 3 *Compute*

$$\begin{aligned}
\mathsf{M}_q &= \begin{bmatrix} R_{0yv}^T & R_{1yv}^T & \cdots & R_{qyv}^T \end{bmatrix}^T V^{-1}, \\
\mathsf{M}_{q-1} &= \begin{bmatrix} I_{n_y(q-1)} & 0 \end{bmatrix} \mathsf{M}_q, \\
\mathsf{O}_{q-1} &= \begin{bmatrix} I_{n_y(q-1)} & 0 \end{bmatrix} \mathsf{O}_q, \\
\mathsf{N}_{q-1} &= \begin{bmatrix} 0 & I_{n_y(q-1)} \end{bmatrix} \mathsf{O}_q.
\end{aligned}$$

let $(\hat{A}, \hat{B}, \hat{C}, \hat{D})$ *be the system matrix of all q Markov Covers of* $\mathcal{T}$, *then*

$$\hat{B} = \tilde{B}(V + \Psi)^{-\frac{1}{2}}$$

and

$$\begin{bmatrix} \hat{D} & \hat{C} \\ \tilde{B} & \hat{A} \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & \mathsf{O}_{q-1}^+ \end{bmatrix} [\mathsf{M}_q \;\; \mathsf{O}_q] + \begin{bmatrix} 0 \\ V_b\hat{U}V_d^T \end{bmatrix}$$

where $\hat{U}$ is an arbitrary matrix satisfying $\hat{U}\hat{U}^T = I$, and V_b and V_a are computed from the SVD's

$$\begin{aligned} \mathsf{O}_{q-1} &= \begin{bmatrix} U_a & U_b \end{bmatrix} \begin{bmatrix} \Sigma_a & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V_a^T \\ V_b^T \end{bmatrix} \\ \begin{bmatrix} \mathsf{M}_{q-1} & \mathsf{N}_{q-1} \end{bmatrix} &= \begin{bmatrix} U_a & U_b \end{bmatrix} \begin{bmatrix} \Sigma_a & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V_c^T \\ V_d^T \end{bmatrix}. \end{aligned}$$

Theorem 3.2: *Suppose a* $\mathrm{QMC_{fsn}}$ *exists for a given data set* Data_q*, then the above* $\mathrm{QMC_{fsn}}$ *algorithm parameterizes all* $\mathrm{QMC_{fsn}}$.
Proof: The proof is similar to the standard q-Markov Cover result in [5] and is not repeated here. □
Remark: The $\mathrm{QMC_{fsn}}$ algorithm provides a q-Markov Cover when the noise is constant $\Xi = \Xi_0$, $\Theta = \Theta_0$ as well in the FSN case. □

4 PLANT IDENTIFICATION

After using the $\mathrm{QMC_{fsn}}$ algorithm to find a linear model with FSN structure for the *closed loop* system, the next task is to extract a model of the plant, given this closed loop model and *a priori* knowledge of the controller. Several ways have been proposed to do this. Method [4] simply subtracts from the closed loop model the known controller dynamics. This leads to a state model of order of the identified closed loop system plus the order of the controller (that is equal to the plant order plus twice the controller order). Hence, model reduction is required to reduce the augmented system to a minimal realization of the plant. This produces a plant model. The approach herein yields a plant representation that is of order of the identified closed loop system (of order of the plant plus controller), and hence is simpler than [4]. This approach to extract the plant from the closed loop model first appears in [15], where both the plant and the controller are assumed to be linear systems. While this plant representation is of lower order than [4], it must still be reduced by model reduction to get a minimal realization of the plant.

Consider the asymptotically stable closed loop system $\mathcal{T}(\mathcal{P},\mathcal{K})$ depicted in Fig. 3. Since $\mathcal{T}(\mathcal{P},\mathcal{K})$ represents the real closed loop system, no mathematical model can capture all properties of $\mathcal{T}(\mathcal{P},\mathcal{K})$. However, it is possible to find a linear model to capture *certain* properties of $\mathcal{T}(\mathcal{P},\mathcal{K})$. For example, for q small enough, it is always possible to obtain a q-Markov Cover for $\mathcal{T}(\mathcal{P},\mathcal{K})$. Suppose $\hat{\mathcal{T}}(\mathcal{P},\mathcal{K})$ is a q-Markov Cover for the closed loop system. We want to construct a linear state space model for the real plant $\mathcal{P}$ based on $\hat{\mathcal{T}}(\mathcal{P},\mathcal{K})$. Since $\hat{\mathcal{T}}(\mathcal{P},\mathcal{K})$

is a linear system with interconnection in Fig. 2, there must exist a linear system $\hat{\mathcal{P}}$ and $\hat{\mathcal{K}}$ such that the transfer function matrix of $\hat{\mathcal{T}}(\mathcal{P},\mathcal{K})$ satisfies

$$\hat{\mathcal{T}}(\mathcal{P},\mathcal{K})(s) = \begin{bmatrix} I & -\hat{\mathcal{K}}(s) \\ -\hat{\mathcal{P}}(s) & I \end{bmatrix}^{-1}. \tag{8}$$

Assume $\hat{\mathcal{T}}(\mathcal{P},\mathcal{K})$ is obtained by using the identification scheme described in the last section and its system matrix is

$$\begin{bmatrix} \hat{D} & \hat{C} \\ \hat{B} & \hat{A} \end{bmatrix}.$$

Partition $\hat{B}$, $\hat{C}$ and $\hat{D}$ according to the dimensions of v_c, v_z, u and z and rewrite the system matrix of $\hat{\mathcal{T}}(\mathcal{P},\mathcal{K})$ as

$$\begin{bmatrix} \hat{D}_{11} & \hat{D}_{12} & \hat{C}_1 \\ \hat{D}_{21} & \hat{D}_{22} & \hat{C}_2 \\ \hat{B}_1 & \hat{B}_2 & \hat{A} \end{bmatrix}, \tag{9}$$

and compute the following matrix

$$\begin{bmatrix} \hat{D}_p & \hat{C}_p \\ \hat{B}_p & \hat{A}_p \end{bmatrix} = \begin{bmatrix} \hat{D}_{21} & \hat{C}_2 \\ \hat{B}_1 & \hat{A} \end{bmatrix} \begin{bmatrix} \hat{D}_{11} & \hat{C}_1 \\ 0 & I \end{bmatrix}^{-1}. \tag{10}$$

Then the linear system $\hat{\mathcal{P}}$ satisfying (8) with FSN structure can be described by

$$\begin{bmatrix} z(k) \\ x(k+1) \end{bmatrix} = \begin{bmatrix} \hat{D}_p & I & \hat{D}_p & \hat{C}_p \\ \hat{B}_p & 0 & \hat{B}_p & \hat{A}_p \end{bmatrix} \begin{bmatrix} \xi(k) \\ \theta(k) \\ u(k) \\ x(k) \end{bmatrix} \tag{11}$$

and the covariances of ξ and θ satisfy assumption (A2).

In this paper, we take the linear system $\hat{\mathcal{P}}$ described in (11) as our model for control design. Note that this model is determined from closed loop data of FSN structure.

Remark: We are only interested in a plant model which is good for control design. Hence whether $\hat{\mathcal{P}}$ is close to $\mathcal{P}$ is not our concern here, since it is known that the open loop error $\hat{\mathcal{P}} - \mathcal{P}$ and the closed loop error $\hat{\mathcal{T}}(\mathcal{P},\mathcal{K}) - \mathcal{T}(\mathcal{P},\mathcal{K})$ may be quite different from each other (small closed loop error and infinite open loop error are not contradictory events. See preface of this special issue). □

5 OUTPUT FEEDBACK CONTROL DESIGN FOR SYSTEM WITH FSN NOISES

Since the identified model $\hat{\mathcal{P}}$ has FSN structure, we must now provide a control method for models with FSN structure. In [2], for continuous time plants, an output feedback controller is found which is robust to FSN noise. This section will provide new characterizations of FSN stability and modifications to [2], and solve the discrete-time FSN control problem. Secondly, the signal-to-noise ratios δ_i^{-1} are assumed to be uncertain parameters falling within certain intervals, and a controller guaranteeing output variance performance over this set of variations of δ_i is given. The measure obtained here is denoted as $\mu_{\rm fsn}$, which is a function of the plant and its controller.

5.1 Stability and Performance Analysis

Let $w(k) \in \mathbb{R}^l$ denote a zero-mean white noise; and $y_i(k) \in \mathbb{R}$ for $i = 1, 2, \cdots, l$. The discrete time, asymptotically stable system $\mathcal{T}$ of l outputs considered here has the following state space description

$$\begin{bmatrix} y_1(k) \\ y_2(k) \\ \cdots \\ y_l(k) \\ x(k+1) \end{bmatrix} = \begin{bmatrix} D_1 & C_1 \\ D_2 & C_2 \\ \cdots & \cdots \\ D_l & C_l \\ B & A \end{bmatrix} \begin{bmatrix} w(k) \\ x(k) \end{bmatrix} \tag{12}$$

Denote the solution of the following Lyapunov equation as L_i

$$L_i = A^T L_i A + C_i^T C_i \tag{13}$$

Since A is a discrete-time stable matrix, then $L_i = L_i^T \geq 0$. Define the set of positive covariances by

$$\mathsf{P} = \{W \; : \; W \in \mathbb{R}^{l\times l}, \;\; W > 0\},$$

then the covariances of all possible input signals $w(k)$ will lie within P.

Definition 5.1: *Let W be the covariance of the input signal $w(k)$ and $y_i(k)$ be the corresponding response of (12) to $w(k)$. For this given W define the variance of the white noise response by*

$$\boldsymbol{V}_i = \mathcal{E}_\infty y_i^2(k), \;\; i = 1, 2, \cdots, l$$

A linear mapping $\boldsymbol{V} \; : \; \mathsf{P} \to \mathbb{R}_+^l$ is called an I/O variance mapping associated *with the LTI system of (12), which could be expressed as*

$$\begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_l \end{bmatrix} = \begin{bmatrix} \boldsymbol{V}_1(W) \\ \boldsymbol{V}_2(W) \\ \vdots \\ \boldsymbol{V}_l(W) \end{bmatrix} = \boldsymbol{V}(W), \quad W \in \mathsf{P}.$$

$\boldsymbol{V}(W)$ can be expressed in terms of the Lyapunov matrices computed from (13)

$$\boldsymbol{V}_i(W) = \mathrm{tr}\{W(D_i^T D_i + B^T L_i B)\}, \quad i = 1, 2, \cdots, l.$$

Now consider the case where $w(k)$ in the above system is an FSN noise. Here assume $w(k) = \psi_0(k) + \psi(k)$. If we denote the output variance of $y_i(k)$ as Y_i for $i = 1, 2, \cdots, l$, and covariances of $\psi_0(k)$ and $\psi(k)$ as $\Psi_0 \in \mathsf{P}$ and $\Psi \in \mathsf{P}$ respectively, then the FSN model implies

$$\Psi = \Delta\,\mathrm{diag}(Y_1, Y_2, \cdots, Y_l) \tag{14}$$

where

$$\Delta = \mathrm{diag}(\delta_1,\ \delta_2,\ \cdots,\ \delta_l).$$

The δ_j's reflect the noise-to-signal ratios satisfying assumption (A2). In this paper, this Δ matrix is assumed to lie within the set

$$\boldsymbol{\Delta} = \{\mathrm{diag}(\delta_1,\ \delta_2,\ \cdots,\ \delta_l) \mid 0 \le \delta_i \le \delta_i^+,\ i = 1, 2, \cdots, l\} \tag{15}$$

where $0 \le \delta_i^+ \le 1$ for $i = 1, 2, \cdots, l$. In this case

$$Y = [Y_1,\ Y_2,\ \cdots,\ Y_l]^T \in \mathbb{R}_+^l$$

will satisfy

$$Y = \boldsymbol{V}(\Psi_0 + \Psi) = \boldsymbol{V}(\Psi_0 + \Delta\,\mathrm{diag}(Y)) \tag{16}$$

In order to emphasize the vector argument Y in the following discussion, a notational abuse is used, where $\boldsymbol{V}(\cdot)$ is taken as a linear mapping from $\mathbb{R}_+^l$ to $\mathbb{R}_+^l$. Using this (abused) notation, (16) simply implies that Y is a fixed point of $\boldsymbol{V}(\cdot)$ in $\mathbb{R}_+^l$. Notice that $\boldsymbol{V}(\cdot)$ is also a function of Δ. However for notational simplicity, we do not explicitly address this dependence which can be determined from the context.

Definition 5.2: *The discrete-time system (12) is said to be robustly FSN stable with respect to* $\boldsymbol{\Delta}$*, if for any* $\Delta \in \boldsymbol{\Delta}$*, there exists a unique and finite vector* $Y \in \mathbb{R}_+^l$ *satisfying (16).*

The following theorem addresses this fixed point or robust FSN stability.

Theorem 5.3: *The following statements are equivalent:*

(i) *(12) is robust FSN stable with respect to* $\boldsymbol{\Delta}$.

(ii) $\forall\, \Delta \in \boldsymbol{\Delta}$, $\boldsymbol{V}(\cdot)\ :\ \mathrm{I\!R}^l_+ \to \mathrm{I\!R}^l_+$ *is a contraction in* $\mathrm{I\!R}^l_+$, *i.e.,* $\forall\, Y, \hat{Y} \in \mathrm{I\!R}^l_+$,

$$\|\boldsymbol{V}(\Psi_0 + \Delta\,\mathrm{diag}(Y)) - \boldsymbol{V}(\Psi_0 + \Delta\,\mathrm{diag}(\hat{Y}))\| < \|\tilde{Y}\|, \tilde{Y} \triangleq Y - \hat{Y}$$

for some vector norm $\|\cdot\|$ *equipped by* $\mathrm{I\!R}^l$.

(iii) $\mu_{\mathrm{fsn}}(\mathcal{T}, \boldsymbol{\Delta}) < 1$, *where*

$$\mu_{\mathrm{fsn}}(\mathcal{T}, \boldsymbol{\Delta}) = \max_{\Delta \in \boldsymbol{\Delta}} \bar{\lambda}(\mathtt{G}\Delta),\ \ \mathtt{G} = \{\mathtt{G}_{ij}\},\ \ \mathtt{G}_{ij} = \boldsymbol{V}(E_j).$$

If the above hold, for any $\Delta \in \boldsymbol{\Delta}$, *the unique fixed point of* $\boldsymbol{V}(\cdot)$ *or the output variance of (12) with FSN noises can be computed as*

$$Y = (I - \mathtt{G}\Delta)^{-1}\boldsymbol{V}(\Psi_0).$$

Proof: See appendix A.

The μ_{fsn} measure in theorem 5.3 can be used to test whether (12) is robustly FSN stable. This μ_{fsn} measure is similar to μ_2 proposed in [2] except that in [2] fixed Δ is considered and μ_{fsn} is characterized by I/O variance mapping.

Now we consider the FSN performance of system $\tilde{\mathcal{T}}$ which is augmented from $\mathcal{T}$ by adding a performance variable $y_0(k)$

$$\begin{bmatrix} y_0(k) \\ y_1(k) \\ y_2(k) \\ \cdots \\ y_l(k) \\ x(k+1) \end{bmatrix} = \begin{bmatrix} D_0 & C_0 \\ D_1 & C_1 \\ D_2 & C_2 \\ \cdots & \cdots \\ D_l & C_l \\ B & A \end{bmatrix} \begin{bmatrix} w(k) \\ x(k) \end{bmatrix}. \tag{17}$$

In FSN noise case, $w(k) = \psi_0(k) + \psi(k)$.

The variance mapping which maps $\Psi_0 + \Psi$ to $\mathcal{E}_\infty\, y_0^T(k)y_0(k)$ is denoted as $\boldsymbol{V}_0$, which can be expressed as

$$\mathcal{E}_\infty\, y_0^T(k)y_0(k) = \boldsymbol{V}_0(\Psi_0 + \Psi) = \mathrm{tr}\{(\Psi_0 + \Psi)(D_0^T D_0 + B^T L_0 B)\}$$

with L_0 satisfying (13).

Definition 5.4: *(17) has robust FSN performance if it is robustly FSN stable and for a given* $\gamma \in \mathrm{I\!R}_+$ *and for all* $\Delta \in \boldsymbol{\Delta}$, $\mathcal{E}_\infty\, y_0(k)^T y_0(k) < \gamma$.

From this definition, the output variance mapping $\boldsymbol{V}_0(\cdot)$ must satisfy

$$\max_{\Delta \in \boldsymbol{\Delta}} \{\boldsymbol{V}_0(\Psi_0 + \Delta\,\mathrm{diag}(Y))\ |\ \ Y \text{ satisfies (16) } \} < \gamma$$

Let

$$Y_0 = \boldsymbol{V}_0(\Psi_0 + \Delta\,\mathrm{diag}(Y))$$

with Y satisfying (16), then we have

$$\begin{aligned} Y_0 &= \boldsymbol{V}_0(\Psi_0) + \boldsymbol{V}_0(\Delta\,\mathrm{diag}(Y)) \\ &= \boldsymbol{V}_0(\Psi_0) + \sum_{i=1}^{l} \boldsymbol{V}_0(\delta_i E_i Y_i) \\ &= \boldsymbol{V}_0(\Psi_0) + \sum_{i=1}^{l} \delta_i \boldsymbol{V}_0(E_i\{(I - \mathtt{G}\Delta)^{-1}\boldsymbol{V}(\Psi_0)\}_i) \end{aligned}$$

where $\{\cdot\}_i$ denotes the i-th element of a vector $\{\cdot\}$. If we further denote

$$\begin{aligned} \tilde{\mathtt{G}} &= \begin{bmatrix} \tilde{\mathtt{G}}_{11} & \tilde{\mathtt{G}}_{12} \\ \tilde{\mathtt{G}}_{21} & \tilde{\mathtt{G}}_{22} \end{bmatrix} \\ \tilde{\mathtt{G}}_{11} &= \boldsymbol{V}_0(\Psi_0) \\ \tilde{\mathtt{G}}_{12} &= \begin{bmatrix} \boldsymbol{V}_0(E_1) & \cdots & \boldsymbol{V}_0(E_l) \end{bmatrix} \\ \tilde{\mathtt{G}}_{21} &= \boldsymbol{V}(\Psi_0) \\ \tilde{\mathtt{G}}_{22} &= \{\mathtt{G}_{ij}\}, \quad \mathtt{G}_{ij} = \boldsymbol{V}_i(E_j) \end{aligned} \tag{18}$$

then Y_0 could be rewritten in the linear fractional form (LFT)

$$Y_0 = \tilde{\mathtt{G}}_{11} + \tilde{\mathtt{G}}_{12}\Delta(I - \tilde{\mathtt{G}}_{22}\Delta)^{-1}\tilde{\mathtt{G}}_{21}.$$

For system $\tilde{\mathcal{T}}$ described by (17), consider the following set

$$\boldsymbol{\Delta}_\gamma = \{\mathrm{diag}(()\delta_0, \Delta) \mid 0 \le \delta_0 \le \frac{1}{\gamma},\ \Delta \in \boldsymbol{\Delta}\}$$

We have the following theorem.
Theorem 5.5: *The following statements are equivalent:*

(i) *(12) has robust FSN performance.*

(ii) $\mu_{\mathrm{fsn}}(\tilde{\mathcal{T}}, \boldsymbol{\Delta}_\gamma) < 1$.

Proof: See appendix A.
Because of the above theorem, if there is an $\alpha \in \mathbb{R}_+$ such that

$$\mu_{\mathrm{fsn}}(\tilde{\mathcal{T}}, \boldsymbol{\Delta}_\gamma) < 1/\alpha$$

then

$$\max_{\Delta \in \alpha\boldsymbol{\Delta}} \{\boldsymbol{V}_0(\Psi_0 + \Delta\,\mathrm{diag}(Y)) \mid Y \text{ satisfies (16)}\} < \gamma/\alpha \tag{19}$$

where

$$\alpha\mathbf{\Delta} = \{\mathrm{diag}(\delta_1, \cdots, \delta_l) \mid 0 \le \delta_i \le \alpha\delta_i^+, i = 1, \cdots, l\}$$

Inequality (19) implies a trade-off between the size of $\mathbf{\Delta}$ and the robust FSN performance.

Now we consider the computation of $\mu_{\mathrm{fsn}}(\tilde{\mathcal{T}}, \mathbf{\Delta}_\gamma)$.

Theorem 5.6: $\mu_{\mathrm{fsn}}(\tilde{\mathcal{T}}, \mathbf{\Delta}_\gamma)$ *can be computed from the following*

$$\mu_{\mathrm{fsn}}(\tilde{\mathcal{T}}, \mathbf{\Delta}_\gamma) = \bar{\lambda}(\tilde{\mathtt{G}}\tilde{\Delta}^+)$$

where

$$\tilde{\Delta}^+ = \begin{bmatrix} \frac{1}{\gamma} & & & \\ & \delta_1^+ & & \\ & & \ddots & \\ & & & \delta_l^+ \end{bmatrix} \tag{20}$$

Proof: See appendix.

5.2 Controller Design for System with FSN Noises

Now we consider the control design for the FSN plant model. For the system obtained from previous identification procedure, the following setup is used

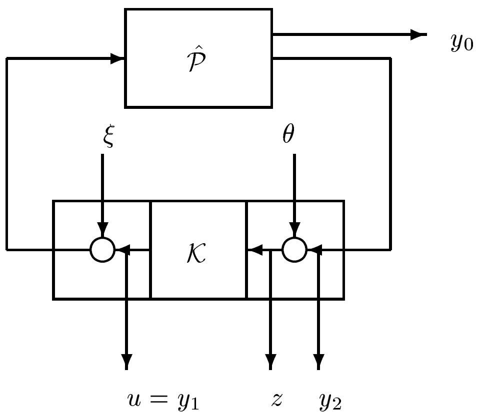

Fig. : 4 The System Setup for Control Design.

where $\hat{\mathcal{P}}$ is the design model (which is a minimal realization of the identified plant model). The state space description of the design model $\hat{\mathcal{P}}$ is

$$\begin{bmatrix} z(k) \\ x_p(k+1) \end{bmatrix} = \begin{bmatrix} \hat{D}_p & \hat{C}_p \\ \hat{B}_p & \hat{A}_p \end{bmatrix} \begin{bmatrix} u(k) + \xi_0(k) + \xi(k) \\ x(k) \end{bmatrix}. \tag{21}$$

The performance variable is defined as

$$y_0(k) = C_0 x(k) + \begin{bmatrix} D_{0_1} & D_{0_2} \end{bmatrix} \begin{bmatrix} \xi_0(k) + \xi(k)) \\ \theta_0(k) + \theta(k) \end{bmatrix}.$$

The realization of the controller $\mathcal{K}$ is denoted as (A_c, B_C, C_c, D_c). Hence

$$\begin{bmatrix} u(k) \\ x_c(k+1) \end{bmatrix} = \begin{bmatrix} D_c & C_c \\ B_c & A_c \end{bmatrix} \begin{bmatrix} z(k) + \theta_0(k) + \theta(k) \\ x_c(k) \end{bmatrix}. \tag{22}$$

The ξ_k in the plant and the $\theta(k)$ in the controller are both FSN noises. The augmented design model which includes the FSN structure and the performance variable can be expressed as

$$\begin{bmatrix} y_0(k) \\ y_1(k) \\ y_2(k) \\ z(k) \\ x_p(k+1) \end{bmatrix} = \begin{bmatrix} D_{0_1} & D_{0_2} & 0 & C_0 \\ 0 & 0 & I & 0 \\ D_p & 0 & D_p & C_p \\ D_p & I & D_p & C_p \\ B_p & 0 & B_p & A_p \end{bmatrix} \begin{bmatrix} \xi_0(k) + \xi(k) \\ \theta_0(k) + \theta(k) \\ u(k) \\ x(k) \end{bmatrix} \tag{23}$$

where

$$\mathcal{E}_\infty \, \xi_i^2(k) = \delta_{c_i} \, \mathcal{E}_\infty \, y_{1_i}^2(k)$$
$$\mathcal{E}_\infty \, \theta_i^2(k) = \delta_{z_i} \, \mathcal{E}_\infty \, y_{2_i}^2(k).$$

Denote $\psi = \begin{bmatrix} \xi^T & \theta^T \end{bmatrix}^T$ and the closed loop system descriptions

$\mathcal{T}_0(\hat{\mathcal{P}}, \mathcal{K})$ = closed loop system descriptions between ψ and y_0,
$\mathcal{T}(\hat{\mathcal{P}}, \mathcal{K})$ = closed loop system descriptions between ψ and $[y_1^T \; y_2^T]^T$,
$\tilde{\mathcal{T}}(\hat{\mathcal{P}}, \mathcal{K})$ = closed loop system descriptions between ψ and $[y_0^T \; y_1^T \; y_2^T]^T$.

Denote the system matrix of $\tilde{\mathcal{T}}(\hat{\mathcal{P}}, \mathcal{K})$ as

$$\begin{bmatrix} \tilde{D}_0 & \tilde{C}_0 \\ \tilde{D}_1 & \tilde{C}_1 \\ \tilde{D}_2 & \tilde{C}_2 \\ \tilde{B} & \tilde{A} \end{bmatrix}.$$

The performance sought here is

$$\mathcal{E}_\infty \, y_0^T(k) y_0(k) = \boldsymbol{V}_0(\Psi_0 + \Delta \, \mathrm{diag}(Y)) < \gamma$$

with Y satisfying

$$Y = \boldsymbol{V}(\Psi_0 + \Delta \, \mathrm{diag}(Y))$$

and $\gamma \in \mathbb{R}_+$ is a prescribed number.

Robust FSN Control Problem: *Given a performance level $\gamma \in \mathbb{R}_+$, find a robust FSN controller such that for all $\Delta \in \mathbf{\Delta}$, the output variance satisfies*

$$\mathcal{E}_\infty\, y_0^T(k) y_0(k)) < \gamma.$$

This problem can be recast into conditions involving μ_{fsn} measure. The optimal robust FSN controller solves

$$\inf_{\mathcal{K}} \mu_{\mathrm{fsn}}(\tilde{\mathcal{T}}(\hat{\mathcal{P}}, \mathcal{K}), \mathbf{\Delta}_\gamma),$$

and the suboptimal robust FSN controller $\mathcal{K}$ solves the following feasibility problem

$$\mu_{\mathrm{fsn}}(\tilde{\mathcal{T}}(\hat{\mathcal{P}}, \mathcal{K}), \mathbf{\Delta}_\gamma) < 1.$$

If no such a controller exists for the given performance level γ, i.e.

$$\inf_{\mathcal{K}} \mu_{\mathrm{fsn}}(\tilde{\mathcal{T}}(\hat{\mathcal{P}}, \mathcal{K}), \mathbf{\Delta}_\gamma) \not< 1,$$

find another performance level $\gamma_1 > \gamma$ and a FSN controller such that γ_1 bounds the FSN output variance.

Since $\mu_{\mathrm{fsn}}(\cdot)$ is a nonlinear function of the controller parameters, no exact solution of the above problem is available. An iterative procedure similar to the one proposed in [2] will now be developed. We will call our algorithm the μ_{fsn} *control design algorithm.*

If we have

$$\mu_{\mathrm{fsn}}(\tilde{\mathcal{T}}(\hat{\mathcal{P}}, \mathcal{K}), \mathbf{\Delta}_\gamma) = \beta,$$

then

$$\beta = h^T \tilde{\mathtt{G}} \tilde{\Delta}^+ d \tag{24}$$

where h and $d = [d_0, d_1, \cdots, d_l]^T$ are the right and left eigenvectors of $\tilde{\mathtt{G}}\tilde{\Delta}^+$ which are normalized to satisfy

$$h^T d = 1, \tag{25}$$

and $\tilde{\Delta}^+$ is defined in (20). From the Perron-Frobenius theorem [19], $h, d \in \mathbb{R}_+^{l+1}$. Partition h according to the dimension 1, n_u and n_z as

$$h = [h_0, h_1^T, h_2^T]^T.$$

Then the right side of the equality (24) is the output variance of the scaled system $\tilde{\mathcal{T}}_h(\hat{\mathcal{P}}, \mathcal{K})$ whose system matrix is denoted as

$$\begin{bmatrix} h_0 \tilde{D}_0 & h_0 \tilde{C}_0 \\ h_1 \tilde{D}_1 & h_1 \tilde{C}_1 \\ h_2 \tilde{D}_2 & h_2 \tilde{C}_2 \\ \tilde{B} & \tilde{A} \end{bmatrix}$$

with respect to the input covariance

$$W(d) = \Psi_0 \delta_0^+ d_0 + \mathrm{diag}(\delta_1^+ d_1, \cdots, \delta_l^+ d_l) \tag{26}$$

i.e., the output variance of $\tilde{\mathcal{T}}_h(\hat{\mathcal{P}}, \mathcal{K})$ satisfies

$$\beta = \boldsymbol{V}_0(W(d)) \tag{27}$$

where $\boldsymbol{V}_0$ is the output variance with respect to the scaled closed loop system $\tilde{\mathcal{T}}_h(\hat{\mathcal{P}}, \mathcal{K})$.

The above discussion is used to to derive the following algorithm, which solves the robust FSN control problem by iterating between the μ_{fsn} measure and the controller $\mathcal{K}$.

The μ_{fsn} Control Design Algorithm:

(i) *Choose $h, d \in \mathbb{R}_+^{l+1}$ satisfying (25). Formulate the scaled plant $\hat{\mathcal{P}}_h$ of the following system matrix*

$$\begin{bmatrix} D_{11} & D_{12} & C_1 \\ D_{21} & D_{22} & C_2 \\ B_1 & B_2 & A \end{bmatrix}$$

where

$$D_{11} = \begin{bmatrix} h_0 D_{0_1} & h_0 D_{0_2} \\ 0 & 0 \\ h_2 D_p & 0 \end{bmatrix}, \; D_{12} = \begin{bmatrix} 0 \\ h_1 I \\ h_2 D_p \end{bmatrix}, \; C_1 = \begin{bmatrix} h_0 C_0 \\ 0 \\ h_2 C_p \end{bmatrix}$$

$$D_{21} = \begin{bmatrix} D_p & I \end{bmatrix}, \; D_{22} = D_p, \; C_2 = C_p$$

$$B_1 = \begin{bmatrix} B_p & 0 \end{bmatrix}, \; B_2 = B_p, \; A = A_p$$

and compute $W(d)$ from (26). Solve the discrete-time LQG or the variance control problem for the general discrete-time system with respect to the input variance $W(d)$ for the scaled open loop plant $\hat{\mathcal{P}}_h(s)$. Denote the controller thus obtained as $\mathcal{K}$.

(ii) *Compute $\tilde{\mathtt{G}}$ (see (18) for definition) from the closed loop system associated with $\mathcal{K}$ and compute*

$$\beta = \bar{\lambda}(\tilde{\mathtt{G}}\tilde{\Delta}^+)$$

where $\tilde{\Delta}^+$ is defined in (20). Find the left and right eigenvectors $\tilde{h}, \tilde{d} \in \mathbb{R}_+^{l+1}$ associated with the eigenvalue β, which are normalized to satisfy (25).

(iii) *If* $\|h-\tilde{h}\|+\|d-\tilde{d}\|<\epsilon$ *(where* $\|\cdot\|$ *is any vector norm equipped by* $\mathbb{R}^{l+1}$*), set* $\mu_{\mathrm{fsn}}=\beta$ *and formulate the final controller, then stop. Otherwise, set*

$$\lambda\tilde{h}+(1-\lambda)h\rightarrow h$$
$$\lambda\tilde{d}+(1-\lambda)d\rightarrow d$$

and go to step 2, where ϵ *is a given error tolerance and* $\lambda\in[0,\ 1]$*.*

Remark: If μ_{fsn} Control Design algorithm fails to yield $\mu_{\mathrm{fsn}}<1$, then the performance bound γ must be increased and another μ_{fsn} control design iteration is required. Unlike the D−K iteration in μ synthesis of robust control, no convergence of the μ_{fsn} Control Design iteration is guaranteed. However, for the example in section 7, a converged μ_{fsn} Control Design occurs at all 5 iterations. For the 5th model iteration, the μ_{fsn} measure with respect to the iterations is shown in Fig. 11. □

6 INTEGRATED CLOSED-LOOP IDENTIFICATION AND CONTROL

The model obtained from open loop identification may or may not be suitable for control design, since the modeling and control design are not independent problems. This has created quite an interest in closed loop identification [3, 5, 8, 9, 10, 11, 12, 13, 14]. The purpose of this section is to give a new algorithm for integrating the closed loop identification (modeling) problem and the control problem. Our conceptual procedures for integrated Closed-Loop Identification and Control (short for CLIC) are summarized as follows.

Procedures for CLIC

Step 1. *For a given real plant* $\mathcal{P}$*, find an initial model* $\hat{\mathcal{P}}$ *of* $\mathcal{P}$*.* $\hat{\mathcal{P}}$ *could be found from either an open-loop identification, or from the first principles of physics given an idealization of the plant. Based on* $\hat{\mathcal{P}}$*, an initial stabilizing controller* $\mathcal{K}$ *is designed.*

Step 2. *The closed loop system thus obtained is denoted as* $\mathcal{T}(\mathcal{P},\mathcal{K})$*. Identifying the real closed loop system* $\mathcal{T}(\mathcal{P},\mathcal{K})$ *by an identification algorithm, denote the system thus obtained as* $\hat{\mathcal{T}}(\mathcal{P},\mathcal{K})$*.*

Step 3. *Extract an open-loop plant model* $\hat{\mathcal{P}}$ *of the real plant* $\mathcal{P}$ *from the identified closed loop system* $\hat{\mathcal{T}}(\mathcal{P},\mathcal{K})$*. This extraction may provide a model with high dimension. In this case, a model reduction step is performed to obtain a minimal realization of* $\hat{\mathcal{P}}$*.*

Step 4. *Design a new controller $\mathcal{K}$ for $\hat{\mathcal{P}}$ such that certain design feasibility criteria are satisfied with respect to the design model $\hat{\mathcal{P}}$. Using this controller to control the real plant $\mathcal{T}(\mathcal{P},\mathcal{K})$, evaluate the real closed loop system based on an experiment. If the performance criteria of the closed loop system are met, stop. Otherwise use the new $\mathcal{K}$ and go to step 2.*

We propose the $\mathrm{QMC}_{\mathrm{fsn}}$ identification for step 2, and the robust FSN controller for step 4. With these ingredients we call the iterative algorithm the FSN-CLIC algorithm, which combines closed loop FSN identification and FSN controller design. The convergence of this algorithm cannot be proven mathematically because it is based upon real data from the real plant for which no "truth model" is assumed. Numerical experience with a structure control problem is given in section 7. The problem statement is as follows.

The FSN-CLIC problem: *Given a stable system $\mathcal{P}$, the FSN ratio bounds $\delta_i^+(i = 1,\cdots,l)$ and an achievable variance requirement γ. Find a low order controller, by solving a closed loop identification and FSN control problem, such that*

(i) *the real closed loop system is stable;*

(ii) *the admissible size of FSN ratios is maximized with respect to the design model $\hat{\mathcal{P}}$;*

(iii) *the real closed loop system has robust FSN performance with level γ.*

Definition 6.1: *If a given model-based controller solves the FSN-CLIC problem, we say that the design model $\hat{\mathcal{P}}$ used for control design is "validated with respect to the performance level γ". If based on this given model, no such controller solves the FSN-CLIC problem one would say that the model is "invalidated with respect to performance level γ".*

Remark: By these definitions, a model can be *validated* but can not be *invalidated*, since our μ_{fsn} control design algorithm can not prove that no such controller exists. □

FSN-CLIC Algorithm

Step 0 *Let γ be the given performance level. Choose integer q (number of Markov, covariance parameters to be matched); integer n_d (length of the experimental data); real number $\epsilon > 0$ (used in convergence criterion); the FSN ratio set $\boldsymbol{\Delta}$.*

Step 1 *Set $i = 0$. If an analytic description of the plant is available, set $\hat{\mathcal{P}}_i$ as the description of this model and go to step 2. Otherwise, proceed to identify a plant model $\hat{\mathcal{P}}_i$ by the* $\mathrm{QMC}_{\mathrm{fsn}}$ *identification algorithm 3.3.*

Step 2 *Robust FSN Controller Design:*

2a *Do model reduction for $\hat{\mathcal{P}}_i$ to obtain a lower order design model $\hat{\mathcal{P}}_{ir}$ for control design.*

2b *Choose variance performance bound κ for the design model $\hat{\mathcal{P}}_{ir}$. Use the robust FSN Control Design algorithm to obtain a controller $\mathcal{K}_i$ and compute the corresponding μ_{fsn} measure. The ϵ is used as the stopping criterion for the μ_{fsn} iteration.*

2c *If the μ_{fsn} Control Design iteration is successful, i.e.*

$$\mu_{\mathrm{fsn}}(\mathcal{T}(\mathcal{P}_{ir}, \mathcal{K}_i), \mathbf{\Delta}_\kappa) < 1$$

go to step 3. Otherwise, choose another $\kappa_1 > \kappa$, set $\kappa = \kappa_1$ and go to step 2b.[2]

Step 3 *Performance Study:*

Evaluate the controller $\mathcal{K}_i$ with the real plant by white noise excitation and compute the output variance. If

$$\mathcal{E}_\infty\, y_0^T y_0 < \gamma$$

then we say the design model $\hat{\mathcal{P}}_{ir}$ is validated with respect to the performance γ and go to step 6; Otherwise the design model is not validated and go to step 4.

Step 4 *Closed-loop Identification:*

Use $\mathrm{QMC}_{\mathrm{fsn}}$ *identification algorithm to obtain a realization of the closed loop system and denote it as $\hat{\mathcal{T}}(\mathcal{P}_i, \mathcal{K}_i)$.*

Step 5 *Set $i = i + 1$. Partition the system matrix of the identified closed loop system $\hat{\mathcal{T}}(\mathcal{P}, \mathcal{K}_{i-1})$ as in (9). Then the plant model $\hat{\mathcal{P}}_i$ can be extracted from $\hat{\mathcal{T}}(\mathcal{P}, \mathcal{K}_{i-1})$ as in section 4, where the system matrix of the plant model $\hat{\mathcal{P}}_i$ is of the form (10). Here the $\hat{\mathcal{P}}_i$ is of the same order as $\hat{\mathcal{T}}(\mathcal{P}, \mathcal{K}_{i-1})$'s. Go to step 2.*

Step 6 *Get the controller from previous iteration.* **Stop**.

Remark: Although κ in robust FSN controller design is tuned for design model, it is only a tuned design parameter for real plant. For the design model, smaller κ will render a controller which achieves better performance (but not necessarily better for the real plant). Finding an optimal κ for the actual closed loop performance is an iterative task, and Table 1 shows the iteration on κ for the example solved. □

[2] $\mathcal{E}_\infty\, y_0^T y_0 < \kappa$ is required for design model and $\mathcal{E}_\infty\, y_0^T y_0 < \gamma$ is required for the real plant. The rule of step 2 is to find a κ for the design model such that the given $\gamma > 0$ is feasible for the real closed loop system.

7 EXAMPLE

A certain smart structure under development at the Structural Systems and Control Laboratory, Purdue University controls the attitude of a rigid mass at the top of the structure. The model of this structure is of order 36. The three outputs to control are the translations of the rigid top in x, y and z directions. For the purpose of this study the digital simulation (complete with computational errors) on a Sun SPARC station 5 is the *real plant* $\mathcal{P}$. The description of the real plant dynamics is known only to the digital simulation but is unknown to identification, control design and performance evaluation procedures. All three procedures are blind to knowledge of the plant.

This system has FSN structure due to sensor/actuator devices, A/D, D/A conversions and computational error in the controller simulation. The actuators and sensors are numerically implemented in the Sun SPARC station 5 to control the real plant $\mathcal{P}$. The signal to noise ratios of the FSN noises are assumed unknown but bounded. We assume this bound

$$\delta_i^+ = 0.25,\ i = 1, 2, \cdots, l.$$

i.e, the variance of signals is at least 4-times larger than the noise variance. Hence we have

$$\mathbf{\Delta} = \{\mathrm{diag}(\delta_1, \cdots, \delta_l)\ :\ 0 \le \delta_i \le 0.25,\ i = 1, \cdots, l\}.$$

The open loop variance can be computed from the white noise time response of $\mathcal{P}$

$$\{\mathcal{E}_\infty\, y_0^T(k) y_0(k)\}_{ol} = 0.8634.$$

The objective of the FSN-CLIC procedure is to find a model-based controller to satisfy a specified performance bound

$$\mathcal{E}_\infty\, y_0^T(k) y_0(k) < \gamma = 0.66$$

for the actual closed loop system. Note that this is not a very severe performance constraint. If a given model-based controller satisfies this performance constraint then we say that the model is "validated with respect to the specified performance bound γ".

The constants used in the FSN-CLIC are

$$q = 150,\ n_d = 20000,\ \epsilon = 10^{-3}$$

The initial model is identified by applying white noise excitation to $\mathcal{P}$ and using the $\mathrm{QMC_{fsn}}$ algorithm to obtain a model $\hat{\mathcal{P}}_0$ of order 59. The D_q matrix obtained from the plant I/O data presents a set of 59 singular values, marked $\times$ in Fig. 5.

A q-Markov Cover model reduction method [7] is used to reduce the 59th order $\hat{\mathcal{P}}_0$ to the 25th order $\hat{\mathcal{P}}_{0r}$. A robust FSN controller is designed for $\hat{\mathcal{P}}_{0r}$. For the purpose of designing a robust FSN controller, the design model performance bound need to be chosen. As we know, it always exists a robust FSN controller for large enough κ, hence we first choose large κ then decrease κ for the stringent performance required for the real plant. We initially pick $\kappa = 4.0$. This κ yields a successful FSN control for the design model $\hat{\mathcal{P}}_{0r}$ but not for the closed loop performance, see from Table 1. In this first iteration, the closed loop output variance is

$$0.687 > \gamma = 0.660$$

and hence the model is not validated. At the third iteration, we achieve a performance of 0.675. The 4th and 5th iterations continue to achieve better performances, but the design model is not validated until iteration 5, where the closed loop performance

$$\mathcal{E}_\infty \, y_0^T(k) y_0(k) = 0.650 < \gamma = 0.660.$$

iteration index	1	2	3	4	5
order of design model $\hat{\mathcal{P}}_{ir}$	25	26	20	20	18
closed loop performance	0.687	0.684	0.675	0.665	0.656
performance bound κ	4.0	2.0	1.818	1.429	1.0
robustness measure μ_{fsn}	0.251	0.241	0.291	0.328	0.469
control energy	0.067	0.110	0.186	0.319	0.654
model validated	No	No	No	No	Yes

Table 1. Information summary of FSN-CLIC iteration

The chosen order of the design model is dictated by large gaps in the singular values of the D_q matrix. Fig. 5 shows the gaps in the singular value distribution of the D_q matrix used for model order determination. Notice that the closed loop data "+" has a much simpler distribution of singular values than those associated with the plant (marked "o" and "x"). These gaps occur at different places on each iteration. The order of the design model on the 5th iteration is chosen as 18. This 18th order model is validated, whereas the 25th order model on iteration 1 is not validated. Hence model validation does not necessarily require more complex models.

Fig. 6 compares the frequency response of all 5 design models during the iterative FSN-CLIC process. The frequency response of the actual plant is given by the dashed line. Note that a model that is validated by our closed loop criterion (model 5) certainly would not be considered as a good model by an open loop criterion (model 1). Fig. 7 suggests that closed loop performance is better using model 5 as the design model.

The fact that model 5 is validated is consistent with the impulse response comparisons of Fig. 8. Throughout the five iterations of FSN-CLIC algorithm, the controller changes are shown in Fig. 9 for channel $y_1 - u_1$, the validated model 5 leads to a controller that is neither the highest nor the lowest gain of all controller iterations. The closed loop behavior is shown in Fig. 7, where the peak of the first model is reduced by a factor 5, using the validated model 5.

Fig. 10 shows the variation of closed loop performance of the actual system (top figure), the closed loop performance bound predicted for the design model, and the $\mu_{\rm fsn}$ measure of robustness. Performance and robustness are almost always in conflict. To get better real closed loop performance, this model must be improved, not just the knowledge of the error bounds. We define a model quality index as

$$MQI = \frac{\mu_{\rm fsn}}{\mathcal{E}_\infty \, y_0^T y_0},$$

where larger MQI implies better model quality. From Table 1, the model 5 has the largest MQI although this was not our criterion for model validation.

8 CONCLUSION

There is no known relationship between errors of the open loop plant model and errors of the closed loop model. This suggests that control design should be done using closed loop data to extract a model of the plant for controller design improvements. This approach allows the design model to be improved, as opposed to some existing approaches which simply increase the error bound to allow the fixed given plant model to yield a stabilizing controller. Such approaches focus on stability rather than performance. Our focus is performance. we define a model to be *valid for control design* if its model-based controller yields the required performance for the closed loop system. This seems to be the only kind of model validation that is possible, since all properties of the real plant cannot be captured by any mathematical model. The only kind of real plant properties we care about are those that will allow successful closed loop performance. The precise properties of the real plant that are required to capture a successful model for control design remain an open question. This paper proposes an algorithm (called FSN-CLIC) to receive closed loop impulse or white noise response data, and to produce a new controller design. The new controller is based upon a plant model that was extracted from the closed loop data. The algorithm has three components

- An identification, called $\mathrm{QMC}_{\mathrm{fsn}}$, allows actuator and sensor noises with FSN structure, where the noise variance is proportional to the signal variance. The constants of proportionality are called the noise to signal ratios δ_i.
- A procedure to extract a plant model from the closed loop system model is given, which makes no assumption about the controller and plant.
- A control design procedure that guarantees (for the design model) an output variance upper bound, while allowing bounded uncertainty in the noise to signal ratios δ_i (This is called a robust FSN controller).

At the conclusion of these three steps, the model generated in step (i)~(ii) is said to be *validated* for μ_{fsn} *control design* if step (iii) produce a controller yielding a specified variance bound

$$\mathcal{E}_\infty \, y_0^T y_0 < \gamma$$

on the actual white noise response of the real system. Since we work with real data, no necessary and sufficient conditions for model validation are possible. However, each of three steps (i), (ii) and (iii) of procedure FSN-CLIC can guarantee these things: in (i) a linear model $\mathrm{QMC}_{\mathrm{fsn}}$ can be constructed that matches the data set D_q from the real (nonlinear) plant. In (ii) the plant model is constructed that (when driven by the known controllers) exactly matches the identified closed loop model. In (iii) the controller guarantees (for the design model) robust performance

$$\mathcal{E}_\infty \, y_0^T y_0 < \gamma$$

over all noise-to-signal ratios within the given bounds.

The example demonstrates that an 18th order linear model can be *validated for control design* for a high order nonlinear plant when the performance criterion is a variance constraint and the identification procedure contains measurement noise. The model that was validated for control design was found after 5 iterations of FSN-CLIC algorithm, this model gives the *best closed loop* performance and one of the *worst* open loop errors with respect to the real plant. Thus, the example suggests the thesis of the paper that models derived from closed loop information can be quite different from models that would be considered good by an open loop criterion.

APPENDIX

Proof of theorem 3.1: Assume (i) holds, then there exists a linear system of the form

$$\begin{bmatrix} y(k) \\ x(k+1) \end{bmatrix} = \begin{bmatrix} D & C \\ B & A \end{bmatrix} \begin{bmatrix} v(k)+\psi(k) \\ x(k) \end{bmatrix}.$$

The output sequence of this system can be obtained as

$$y_q(k) = \mathsf{O}_q x(k) + \mathsf{H}_q(v_q(k)+\psi_q(k)), \tag{28}$$

where

$$\mathsf{O}_q = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{q-1} \end{bmatrix}, \quad \mathsf{H}_q = \begin{bmatrix} H_0 & 0 & \cdots & 0 \\ H_1 & H_0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ H_{q-1} & H_{q-2} & \cdots & H_0 \end{bmatrix}$$

$$v_q(k) = \begin{bmatrix} v^T(k) & v^T(k+1) & \cdots & v^T(k+q-1) \end{bmatrix}^T$$
$$\psi_q(k) = \begin{bmatrix} \psi^T(k) & \psi^T(k+1) & \cdots & \psi^T(k+q-1) \end{bmatrix}^T$$
$$y_q(k) = \begin{bmatrix} y^T(k) & y^T(k+1) & \cdots & y^T(k+q-1) \end{bmatrix}^T,$$

where $\{H_i \mid i = 0,1,\cdots,q-1\}$ denotes the set of Markov parameters with

$$H_0 = D, \;\; H_i = CA^{i-1}B, \;\; i = 1,2,\cdots,q-1.$$

Considering that $v(k)$ is independent of $\psi(k)$, and taking covariance operation of $y_q(k)$ in (28) yields

$$\mathsf{R}_q = \mathcal{E}_\infty\, y_q(k) y_q(k)^T = \mathsf{O}_q X \mathsf{O}_q^T + \mathsf{H}_q\{I \otimes (V+\Psi)\}\mathsf{H}_q^T \tag{29}$$

with Ψ satisfying (7) and X satisfying

$$X = AXA^T + B(V+\Psi)B^T.$$

Denote $\bar{v}(k) = v(k) + \psi(k)$. From (28), we have

$$\begin{aligned} R_{iyv} &= \mathcal{E}_\infty\, y(k+i)v^T(k) \\ &= \mathcal{E}_\infty (CA^i x(k) + H_i\bar{v}(k) + H_{i-1}\bar{v}(k+1) + \cdots + H_0\bar{v}(k+i))v^T(k) \\ &= H_i\, \mathcal{E}_\infty\, \bar{v}(k)v^T(k) = H_i V. \end{aligned}$$

Hence (29) leads to

$$\mathsf{R}_q - \mathsf{R}_{qyv}\mathsf{V}^{-1}(\mathsf{V} + I \otimes \Psi)\mathsf{V}^{-T}\mathsf{R}_{qyv}^T = \mathsf{O}_q X \mathsf{O}_q^T \geq 0$$

i.e., (i) implies (ii).

On the other hand, if (ii) is satisfied, the $\mathrm{QMC_{fsn}}$ algorithm will provide a liner system which matches the data set Data_q. Hence (ii) implies (i). □

Proof of theorem 5.3: $\forall\ Y, \hat{Y} \in \mathbb{R}^l_+$, define $\tilde{Y} = Y - \hat{Y}$. Consider the difference

$$\begin{aligned}
\boldsymbol{V}(\Psi_0 + \Delta\,\mathrm{diag}(Y)) - \boldsymbol{V}(\Psi_0 + \Delta\,\mathrm{diag}(\hat{Y})) &= \boldsymbol{V}(\Delta\,\mathrm{diag}(\tilde{Y})) \\
&= \boldsymbol{V}(\sum_{i=1}^{l} \delta_i \tilde{Y}_i E_i) \\
&= \sum_{i=1}^{l} \delta_i \tilde{Y}_i \boldsymbol{V}(E_i) \\
&= \mathtt{G}\Delta\tilde{Y}
\end{aligned}$$

(ii) implies that for all $\Delta \in \boldsymbol{\Delta}$

$$\|\mathtt{G}\Delta\tilde{Y}\| \leq \|\tilde{Y}\|,$$

or for all $\Delta \in \boldsymbol{\Delta}$ there exists an induced matrix norm $\|\cdot\|_{in}$ associated with the vector norm $\|\cdot\|$ such that

$$\|\mathtt{G}\Delta\|_{in} < 1.$$

Since the matrix spectral radius is less than any induced matrix norm, the above is equivalent to

$$\max_{\Delta\in\boldsymbol{\Delta}} \rho(\mathtt{G}\Delta) < 1.$$

Consider that $\mathtt{G}\Delta$ is a matrix of all positive elements, by the Perron-Frobenius theorem [19]

$$\rho(\mathtt{G}\Delta) = \bar{\lambda}(\mathtt{G}\Delta).$$

Hence (ii) is equivalent to (iii).

(ii) $\Rightarrow$ (i) is a standard result, hence (iii) $\Rightarrow$ (i). Now we assume that (iii) fails, i.e., there exists a $\Delta_0 \in \boldsymbol{\Delta}$ such that

$$\bar{\lambda}(\mathtt{G}\Delta_0) \not< 1$$

Since $0 \in \boldsymbol{\Delta}$ and $\bar{\lambda}(\mathtt{G}\Delta)$ is a continuous function of Δ, hence there must exist another $\Delta_1 \in \boldsymbol{\Delta}$ such that $\bar{\lambda}(\mathtt{G}\Delta_1) = 1$, or $I - \mathtt{G}\Delta_1$ is singular. This implies that the following equations will not have a unique solution for Y

$$Y = \boldsymbol{V}(\Psi_0) + \mathtt{G}\Delta Y. \tag{30}$$

Hence if (ii) fails, then $\boldsymbol{V}(\cdot)$ does not have a unique solution for Δ_1, i.e., (i) fails. By solving Y from (30), the expression for Y follows. □

Proof of Theorem 5.5: FSN performance implies that (12) is robustly FSN stable or $\forall\ \Delta \in \mathbf{\Delta}$

$$\det(I - \tilde{\mathtt{G}}_{22}\Delta) > 0,$$

and the output variance Y_0 satisfies

$$1 - Y_0 \frac{1}{\gamma} > 0,$$

which is equivalent to for all $0 \leq \delta_0 \leq \frac{1}{\gamma}$, $\Delta \in \mathbf{\Delta}$

$$\det(I - \tilde{\mathtt{G}}_{22}\Delta)[1 - \delta_0(\tilde{\mathtt{G}}_{11} + \tilde{\mathtt{G}}_{12}\Delta(I - \tilde{\mathtt{G}}_{22}\Delta)^{-1}\tilde{\mathtt{G}}_{21})] > 0.$$

This is equivalent to for all $\tilde{\Delta} \in \mathbf{\Delta}_\gamma$

$$\det(I - \tilde{\mathtt{G}}\tilde{\Delta}) > 0.$$

Hence the claim follows. □

Proof of Theorem 5.6: We first prove that

$$\mu_{\text{fsn}}(\tilde{\mathcal{T}}, \mathbf{\Delta}_\gamma) = \max_{\tilde{\Delta} \in \mathbf{\Delta}_\gamma^v} \bar{\lambda}(\tilde{\mathtt{G}}\tilde{\Delta})$$

where $\mathbf{\Delta}_\gamma^v$ is the set of all vertices of $\mathbf{\Delta}_\gamma$. For given $\tilde{\Delta} \in \mathbf{\Delta}_\gamma$ and $\rho > 0$, denote

$$f(\tilde{\Delta}, \rho) = \det(I - \rho\tilde{G}\tilde{\Delta}).$$

Consider the set

$$H(\rho) = \{f(\tilde{\Delta}, \rho) \ : \ \tilde{\Delta} \in \mathbf{\Delta}_\gamma\}.$$

By definition, we have

$$\begin{aligned}
\mu_{\text{fsn}}(\tilde{\mathcal{T}}, \mathbf{\Delta}_\gamma) &= \max_{\tilde{\Delta} \in \mathbf{\Delta}_\gamma} \bar{\lambda}(\tilde{\mathtt{G}}\tilde{\Delta}) \\
&= \max\{\lambda > 0 \ : \ 0 \in H(\frac{1}{\lambda})\} \\
&= \min\{\lambda > 0 \ : \ 0 \notin H(\frac{1}{\lambda})\} \\
&= [\max\{\rho > 0 \ : \ 0 \notin H(\rho)\}]^{-1}
\end{aligned}$$

Since $f(\tilde{\Delta}, \rho)$ is a multilinear function of $\delta_i (i = 0, 1, \cdots, l)$, hence for given $\rho > 0$, $H(\rho)$ is an interval

$$\min_{\tilde{\Delta} \in \mathbf{\Delta}_\gamma^v} f(\tilde{\Delta}, \rho) \leq H(\rho) \leq \max_{\tilde{\Delta} \in \mathbf{\Delta}_\gamma^v} f(\tilde{\Delta}, \rho)$$

where $\mathbf{\Delta}_\gamma^v$ denotes all the vertices of $\mathbf{\Delta}_\gamma$. i.e., for given $\rho > 0$, $0 \notin H(\rho)$ is equivalent to either

$$\min_{\tilde{\Delta}\in\mathbf{\Delta}_\gamma^v} f(\tilde{\Delta},\rho) > 0 \quad \text{or} \quad \max_{\tilde{\Delta}\in\mathbf{\Delta}_\gamma^v} f(\tilde{\Delta},\rho) < 0.$$

Hence we have

$$\mu_{\text{fsn}}(\tilde{\mathcal{T}},\mathbf{\Delta}_\gamma) = \max_{\tilde{\Delta}\in\mathbf{\Delta}_\gamma^v} \bar{\lambda}(\tilde{\mathtt{G}}\tilde{\Delta}).$$

Further by the Perron-Frobenius theorem [19], for any $\tilde{\Delta} \in \mathbf{\Delta}_\gamma^v$, there exist eigenvectors $h, d \in \mathbb{R}_+^{l+1}$ such that

$$\bar{\lambda}(\tilde{\mathtt{G}}\tilde{\Delta}) = h^T\tilde{\mathtt{G}}\tilde{\Delta}d$$

hence

$$\max_{\tilde{\Delta}\in\mathbf{\Delta}_\gamma^v} \bar{\lambda}(\tilde{\mathtt{G}}\tilde{\Delta}) = \bar{\lambda}(\tilde{\mathtt{G}}\tilde{\Delta}^+).$$

This proves theorem 5.6. □

REFERENCES

1. Skelton, R. E.: "Robust Control of Aerospace Systems,"in *IFAC Symposium on Robust Control Design*, pp. 24-31, 1994.
2. Thygesen, U. H., Skelton, R. E.: "Linear System with Finite Signal-to-Noise Ratios: A Robustness Approach," *Proc. 34th IEEE Conf. on Decision & Contr.*, 1995.
3. Skelton, R. E.: "Model Error Concept in Control Design," *Int. J. of Contr.*, Vol. 49 No. 5 pp. 1725-53, 1989.
4. Liu, K., Skelton, R. E.: "Closed-loop Identification and Iterative Controller Design,"*Proc. 29th IEEE Conf. on Decision & Contr.*, 1990.
5. Skelton, R. E., Shi, G.: "A Weighted q-Markov Cover and Iterative Identification and Control," to appear in special issue on "Subspace Methods for Detection and Estimation", Pt. II: System Identification, *EURASIP Signal Processing*, 1996.
6. Zhu, G., Skelton, R. E. and Li, P.:"q-Markov Cover Identification Using Pseudo Random Binary Signals," *Int. J. of Contr.*, 1995.
7. King, A. M., Desai, U. B., Skelton, R. E.: "A Generalized Approach to Covariance Equivalent Realizations for Discrete Systems," *Automatica*, Vol. 24, No. 4, pp. 507-515, 1988.
8. Van den Hof, P. M. J., Schrama, R. J. P.:"Identification and Control - Closed-loop Issues," *10th IFAC Symposium on System Identification*, Vol. 2, pp. 1-14, 1994.
9. Gevers, M.: " Towards a Joint Design of Identification and Control?" In H. L. Trentelman and J. C. Willems, editors, *Essays on Control: Perspectives in the Theory and Its Applications*, Birkhauser, 1993.

10. Lee, W. S., Anderson, B. D. O., Mareels, I. M. Y., Kosut, R. L.: "On Some Key Issues in the Windsurfer Approach to Adaptive Robust Control," *Automatica*, Vol. 31 No. 11 pp. 1619-36, 1995.
11. Hjalmarsson, H., Gevers, M., De Bruyne, F., Leblond, J.:"Identification for Control: Closing the Loop gives More Accurate Controllers," *Proc. 33th IEEE Conf. on Decision & Contr.*, pp. 4150-5, 1994
12. R. G. Hakvoort, Schrama, R. J. P., Van den Hof, P. M. J.: "Approximate identification with closed loop performance criterion and application to LQG feedback design" *Automatica* Vol. 30, No. 4, pp. 679-690, 1994.
13. Zang, Z., Bitmead, R. R., Gevers, M.: "Iterative Weighted Least-squares Identification and Weighted LQG Control Design," *Automatica*, Vol. 31, No. 11, pp. 1577-94, 1995.
14. Van den Hof, P. M. J., Schrama, R. J. P., de Callafon, R. A., Bosgra, O. K.: "Identification of Normalized Coprime Plant Factors from Closed-loop Experimental Data," *European Journal of Control*, Vol. 1, No. 1, pp. 62-74, 1995.
15. A. C. van der Klauw, M. Verhaegen, van den Bosch, P. P. J.: "State Space Identification of Closed Loop Systems," *Proc. 31st IEEE Conf. on Decision & Contr.*, pp. 1327-1332, 1991.
16. Smith, R., Dullerud, G.: "Continuous-Time Control Model Validation Using Finite Experimental Data," To appear in *IEEE Trans Autom Control.*
17. Dullerud, G., Smith, R.: "Sampled Data Model Validation: An Algorithm and Experimental Application," To appear in *Int. J. of Robust and Nonlinear Contr.*
18. Smith, R., Dullerud, G., Rangan, S., Poola, K.: "Model Validation for Dynamically Uncertain Systems," This issue.
19. Horn, R. A., Johnson, C. R.: *Matrix Analysis*, Cambridge University Press, Cambridge, 1985.
20. Gevers, M., Bitmead, R., R., Blondel, V.: "Unstable Ones Understood Algebraic Questions of Modeling for Control Design" This issue.
21. Callafon, R. A., van den Hof, P. M. J.: "Suboptimal Feedback Control by a Scheme of Iterative Identification and Control Design," This issue.
22. Bernstein, D.:"Robust Static and Dynamic Output Feedback Stabilization: Deterministic and Stochastic Perspectives," *IEEE Trans Autom Control*, pp. 1076-1084, 1987.

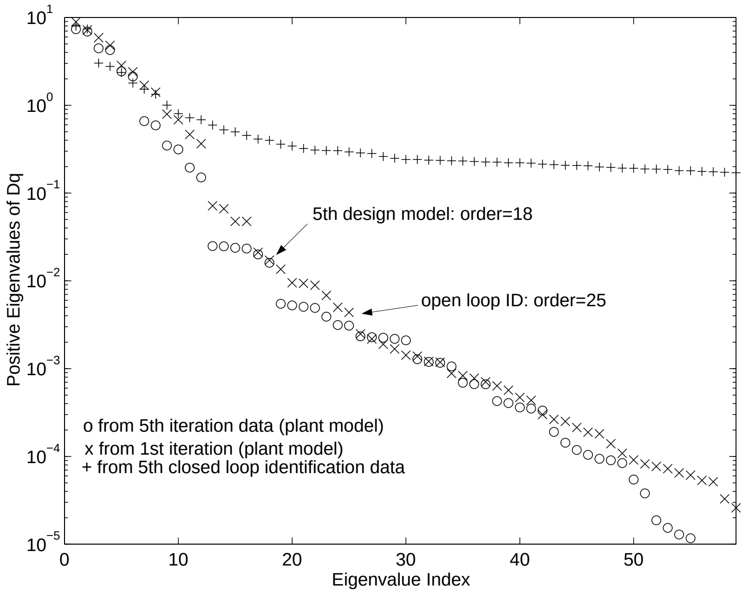

Fig. 5. The singular values of the D_q matrix from the initial identification data, the 5th closed loop data and the 5th open-loop plant data.

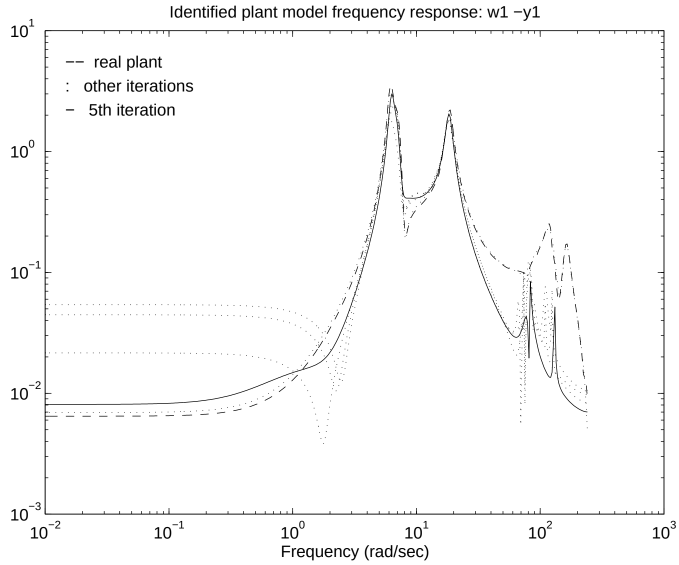

Fig. 6. The Bode plots of the identified plant model transfer function from w_1 to y_1.

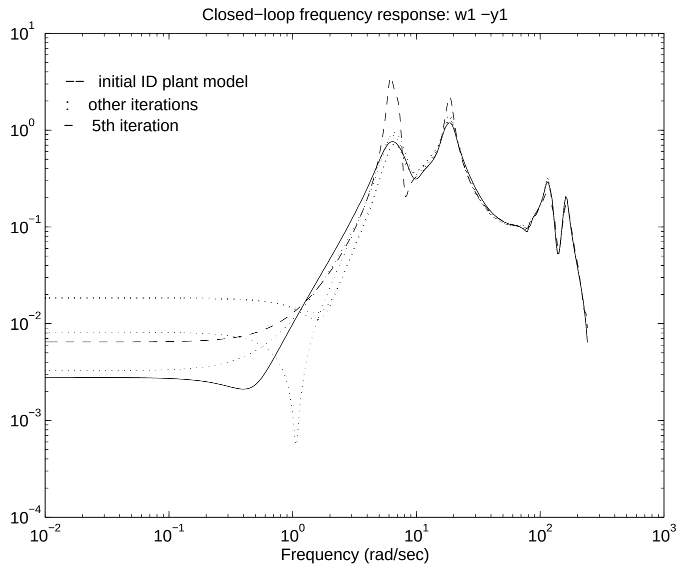

Fig. 7. The Bode plots of the closed loop transfer function from w_1 to y_1 with respect to different iterations.

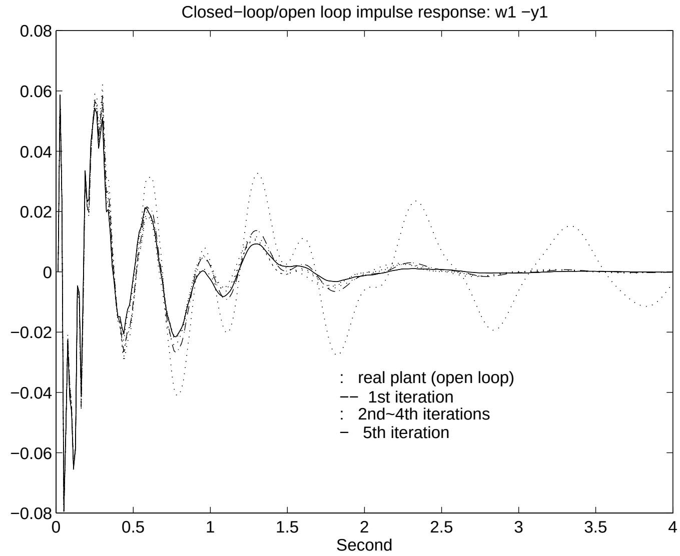

Fig. 8. The impulse response of the closed loop system in different iterations

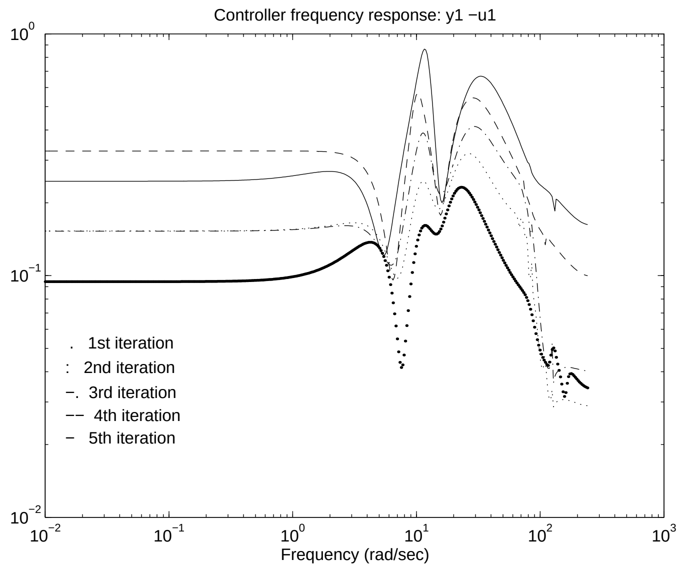

Fig. 9. The Bode plots of the controller transfer function from y_1 to u_1.

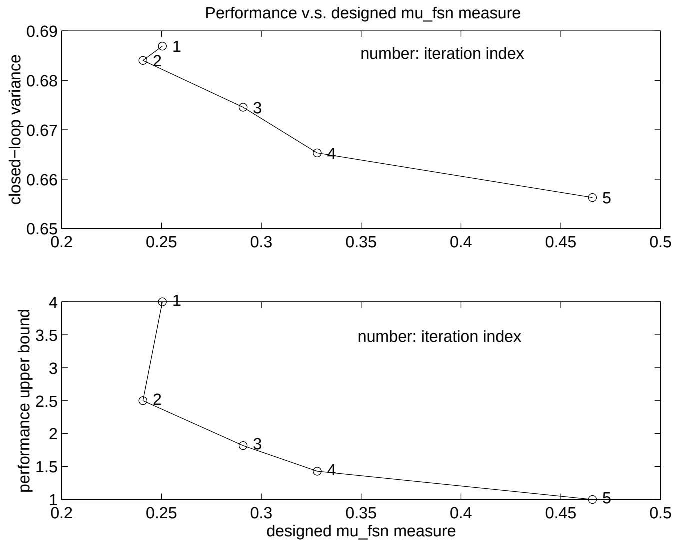

Fig. 10. The closed loop output variance v.s. the designed robustness; the performance upper bound for the controlled design models v.s. the designed $\mu_{\rm fsn}$.

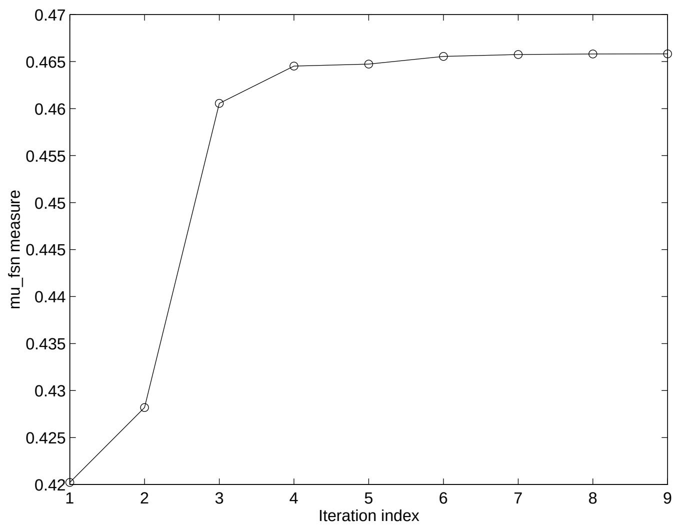

Fig. 11. Iterative values for $\mu_{\rm fsn}$ measure during FSN D-K iteration (the 5th model/controller iteration)