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Integrating instrumentation and control design

JIANBO LU† and ROBERT E. SKELTON‡

Instrument precision is characterized by its signal-to-noise ratio, employing a new noise model. In this paper, this measure of precision of each instrument is used to characterize instrumentation, and an integration is achieved by jointly optimizing the feedback control law and the instrument signal-to-noise ratios to meet control system performance requirements. Iterative algorithms are proposed to find locally optimal solutions. Assuming that the signal-to-noise ratio is directly related to the instrumentation cost, this integration provides a systematic procedure to design a low cost control system. More importantly, this procedure identifies the performance-limiting components of a control system, identifies where to spend money on a system, and generates component design requirements from closed loop system performance criteria.

1. Notation

In this paper $\mathbb{R}_+$ and $\mathbb{R}$ denote the sets of positive real numbers and real numbers respectively. $\mathbb{C}_-$ denotes the set of all complex numbers with their real part strictly less than zero. The n-dimensional linear spaces over $\mathbb{R}_+$ and $\mathbb{R}$ are denoted by $\mathbb{R}_+^n$ and $\mathbb{R}^n$ respectively. $\det(\cdot)$, $\lambda(\cdot)$, $\bar{\lambda}(\cdot)$, $\rho(\cdot)$, $\mathrm{tr}(\cdot)$ and $(\cdot)^{\mathrm{T}}$ denote the determinant, the set of all eigenvalues, the largest eigenvalue, the spectral radius, the trace and the transpose of a matrix $(\cdot)$ respectively. For matrix operations, $\geq$ means semi-positive definite and $\leq$ means semi-negative-definite. $\|\cdot\|$ is the vector Euclidean norm. $\mathbf{E}[\cdot]$ denotes the usual expectation operator of a stochastic variable, $\mathbf{E}_\infty[\cdot]$ is the steady-state value of the expectation of a stochastic process. $[\cdot]_{ij} \triangleq (i,j)$-element of a matrix $[\cdot]$. block diag$(\cdot) \triangleq$ block diagonal augment of matrices.

2. Introduction

Control system integration should begin with a selection of the components (sensors, actuators etc.), but theory is not available to decompose the specified performance of the closed loop system into performance requirements of the components. Today, most control system designs begin with a selection of the components (sensors, actuators etc.) and the architecture is specified (sensor/actuator configuration has been given). However, the choice of feedback control law, the selection of variables to be measured and controlled, and the choice of design specifications for the given hardware are not independent problems. Technology based upon the traditional separation of these problems might not yield performance comparable to joint optimization of control and system parameters. Well-known results show that the system performance might be improved by deleting a noisy actuator, or might be degraded in implementation due to a hardware component with improper design specifications. Today, when performance is not acceptable, it is not clear whether to modify the manufacturing tolerance, the signal processing, the sensor accuracy and/or the control law. Therefore, in order to improve performance, a theory is needed to jointly design the feedback control law and the instrumentation configuration.

Instruments may be characterized by their linear dynamic ranges, bandwidths, signal-to-noise ratios etc. In this paper we consider the limitation of signal-to-noise ratios. Electronic instruments have finite signal-to-noise ratios, usually determined through testing. In many electronic devices, the size of the noise is proportional to the size of the signal. Noise models of this type are key to this paper. Many examples of this kind of finite signal-to-noise effect in electro-mechanical systems can be found in Lu (1997).

Although the finite signal-to-noise ratio defined here is a well-known phenomenon in instrumentation, control system design for this model is very new. Ruan and Choudhury (1993) and Krolikowski (1997) studied the LQG control problem for a first-order system (the state is a scalar) where the actuator noise variance is linearly related to the actuator signal variance. Skelton (1994) was the first to introduce the *finite signal-to-noise* model for general linear systems. For fixed signal-to-noise ratios, designing a feedback control algorithm to tolerate the finite-signal-to-noise effect and to achieve sub-optimal performance requirements is studied in Thygesen and Skelton (1995), Shi and Skelton (1995), Lu *et al.* (1996), Lu and Skelton (1997a) and Skelton and Lu (1997). Lu and Skelton (1997b) proposed a

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† Author for correspondence. Advanced Control Systems, Delphi Chassis Technical Center, General Motors. Mailing address: M/C E520, PO Box 1245, 1435 Cincinnati Street, Dayton, OH 45401, USA. e-mail: lnusdyt1.pzrm03@gmeds.com

‡ Department of Applied Mechanics and Engineering Sciences, University of California at San Diego, La Jolla, CA 92093-0411, USA. e-mail: reskelton@ames.ucsd.edu

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convergent algorithm to synthesize an output feedback controller.

In this paper, we consider the case where both the instrumentation achitecture (configuration) and the control algorithm are subject to design. The *signal-to-noise ratio* is one way to define precision, and the cost and the quality of a hardware instrument will certainly be related to its signal-to-noise ratio. If the cost were linearly related to its signal-to-noise ratio, the integrated design of the signal-to-noise ratio and the feedback control algorithm would help to answer the following question:

Economic design problem: For a given performance requirement, design the feedback control law and distribute signal-to-noise ratios among the instruments (sensors, actuators, A/D, D/A conversion, control processing) such that the instrumentation cost is minimized without compromising the control system performance.

This problem was first proposed in Skelton (1994). In this paper we will investigate the theoretical issues related to the integration and then propose several numerical algorithms to address this problem. No convergence guarantee is given.

This paper is organized as follows. Section 2 provides the mathematical modelling for the continuous-time finite signal-to-noise model and analysis results. Section 3 contains the main results and algorithms of this paper to conduct the design for both signal-to-noise ratios and feedback control laws. Section 4 provides an example. Section 5 gives a brief conclusion.

3. FSN modelling and mean-square analysis

Consider the following differential equation

$$\dot{x} = Ax + \sum_{j=0}^{n} B_j w_j, \quad z_i = C_i x \tag{1}$$

where $x(t) \in \mathbb{R}^{n_x}$ is the state variable. $z_i(t) \in \mathbb{R}^{q_i}$ for $i \in \{0, \dots, n\}$. $w_j(t) \in \mathbb{R}^{p_j}$, $j \in \{0, \dots, n\}$ are white noise processes and $w_0(t)$ is assumed to have unity spectral density. The following augmented variables will be used in the sequential discussion

$$\hat{z} = [z_0^{\mathrm{T}} \quad z^{\mathrm{T}}]^{\mathrm{T}}, \quad \hat{w} = [w_0^{\mathrm{T}} \quad w^{\mathrm{T}}]^{\mathrm{T}}$$

where

$$z = [z_1^{\mathrm{T}} \quad z_2^{\mathrm{T}} \quad \dots \quad z_n^{\mathrm{T}}]^{\mathrm{T}}, \quad w = [w_1^{\mathrm{T}} \quad w_2^{\mathrm{T}} \quad \dots \quad , w_n^{\mathrm{T}}]^{\mathrm{T}}$$

Let the $w_j(t)$s be uncorrelated, white noise processes. Denote $S_{w_j}(j\omega)$ as the spectral density of $w_j(t)$. For the white noise process $w_j(t)$, this spectral density is constant over frequency, i.e. there exists a constant W_j such that

$$S_{w_j}(j\omega) = W_j$$

We consider also the case where $w_j(t)$ is white but its spectral density is characterized only by its upper bound $\pi_j I$

$$S_{w_j}(j\omega) = W_j \leq \pi_j I, \quad j \in \{1, \dots, n\}$$

In addition, if this constant spectral density is proportional to the variance $\mathbf{E}_\infty[\|z\|^2]$ of the signal $z(t)$, then we have the following finite-signal-to-noise model.

Definition 1: Assume w_js in (1) are uncorrelated, white noise processes; we say the system (1) has a finite-signal-to-noise model (FSN model) if the spectral density matrix W_j of w_j satisfies

$$W_j = \mathbf{E}_\infty[\|z_j\|^2]R_j, \quad 0 \leq R_j \leq \sigma_j I, \quad j \in \{1, \dots, n\}$$

Denote

$$\boldsymbol{R}^\sigma \triangleq \{\text{block diag}\,(R_1, R_2, \dots, R_n): \; 0 \leq R_j \leq \sigma_j I, j \in \{1, \dots, n\}\}$$

and the set of all FSN processes $\mathcal{W}(\sigma, z)$, where $\sigma = [\sigma_1 \quad \sigma_2 \quad \dots \quad \sigma_n]^{\mathrm{T}}$. The jth element σ_j of σ is the upper bound of the noise-to-signal ratio for the jth channel; σ_j^{-1} is called the signal-to-noise ratio of the jth channel.

Let ζ_i be the steady-state variance of the system signal $z_i(t)$ in (1) for $i \in \{1, \dots, n\}$. For the above FSN model, the state covariance matrix

$$X = \lim_{t \to \infty} \mathbf{E}[x(t)x^{\mathrm{T}}(t)]$$

of the system (1) depends on ζ_is. On the other hand, ζ_is can be determined from the state covariance matrix X. That is, if in the system (1) w_is obey the FSN model, then there exists an interrrelationship between ζ_js and X. It might be possible that ζ_is are finite and X is infinite, or ζ_is and X are both finite, or ζ_is and X are both infinite. We need to distinguish these cases. In this paper, we are interested in the case where ζ_is and X are both finite. We first consider the following definition.

Definition 2: The system (1) is said to be FSN stable if, for all initial states $x(0)$, the steady state variance

$$\zeta_i = \mathbf{E}_\infty[\|z_i(t)\|^2]$$

exists, and is finite and unique for $i \in \{1, \dots, n\}$.

The finiteness of ζ_js might not imply the finiteness of the X. For a stochastic system (1), mean-square property is a well-used concept to capture the finiteness of X. The following provides a definition for the stability in the mean-square sense, which we will call mean-square stability (MSS).

Definition 3: The system (1) is mean-square stable (MSS) if, for all initial states $x(0)$, the state mean $\mathbf{E}[x(t)]$ converges to zero and the state covariance

$$X(t) = \mathbf{E}[x(t)x^{\mathrm{T}}(t)]$$

converges to a finite value as $t \to \infty$.

If (1) is mean-square stable, then there exists an $X \geq 0$ and $\zeta_i \in \mathbb{R}_+$ for $i \in \{1, \ldots, n\}$ such that

$$0 = AX + XA^{\mathrm{T}} + B_0B_0^{\mathrm{T}} + \sum_{j=1}^{n} \zeta_j B_j R_j B_j^{\mathrm{T}}, \quad \zeta_i = \mathrm{tr}\,[C_i X C_i^{\mathrm{T}}]$$

If (1) is FSN stable, then there exists $\zeta_i \in \mathbb{R}_+$ for $i \in \{1, \ldots, n\}$ such that

$$\zeta_i = \mathrm{tr}\left[\int_0^\infty C_i \mathrm{e}^{A\tau}\left\{B_0B_0^{\mathrm{T}} + \sum_{j=1}^{n} \zeta_j B_j R_j B_j^{\mathrm{T}}\right\}\mathrm{e}^{A^{\mathrm{T}}\tau} C_i^{\mathrm{T}}\, \mathrm{d}\tau\right] \tag{2}$$

Denote $\zeta \triangleq [\zeta_1 \quad \zeta_2 \quad \ldots \quad \zeta_n]^{\mathrm{T}}$; then, from (2), ζ satisfies the following fixed point relationship

$$\zeta = \mathcal{V}(\zeta) \tag{3}$$

for a properly defined function $\mathcal{V}$. The following theorem characterizes the existence conditions for a ζ to satisfy (3), i.e. the FSN stability, which is further proved to be equivalent to mean-square stability.

Theorem 1: *The following statements are equivalent:*

(*i*) (1) *is FSN stable, i.e. there exists a finite and unique ζ satisfying* (2).

(*ii*) $\lambda(A) \in \mathbb{C}_-$ *and* $\mathcal{V}(\cdot)$: $\mathbb{R}_+^n \to \mathbb{R}_+^n$ *is a contraction in* $\mathbb{R}_+^n$, *i.e.* $\forall \zeta, \hat{\zeta} \in \mathbb{R}_+^n$,

$$\|\mathcal{V}(\zeta) - \mathcal{V}(\hat{\zeta})\| < \|\tilde{\zeta}\|$$

for some vector norm $\|\cdot\|$ *in* $\mathbb{R}^n$.

(*iii*) $\lambda(A) \in \mathbb{C}_-$. *Then*

$$\max_{R \in \boldsymbol{R}^\sigma} \bar{\lambda}(\mathtt{G}_R) < 1$$

where the (i,j)*th element of* $\mathtt{G}_R$ *is*

$$[\mathtt{G}_R]_{ij} = \|T_{z_i w_j} R_j^{1/2}\|_2^2$$

and $T_{z_i w_j}$ *is the transfer matrix from* w_j *to* z_i *for* $i,j \in \{1, \ldots, n\}$. *Further, we have*

$$\max_{R \in \boldsymbol{R}^\sigma} \bar{\lambda}(\mathtt{G}_R) = \bar{\lambda}(\mathtt{H}\,\mathrm{diag}\,(\sigma))$$

where the (i,j)*th element of* $\mathtt{H}$ *is* $[\mathtt{H}]_{ij} = \|T_{z_i w_j}\|_2^2$. *We define this quantity as a robustness measure*

$$\phi(T, \sigma) \triangleq \bar{\lambda}(\mathtt{H}\,\mathrm{diag}\,(\sigma))$$

(*iv*) (1) *is mean-square stable.*

Proof: See Appendix. □

The above theorem characterizes the FSN or mean-square stability by a simple mathematical measure ϕ. In the following we would like to extend this to performance characterization. The performance studied in this paper is the maximum output variance of z_0 over the FSN process $\mathcal{W}(\sigma, z)$ for a given set of noise-to-signal ratios σ. If we denote the performance (worst case performance) as J, then

$$J \triangleq \max_{w \in \mathcal{W}(\sigma, z)} \{\mathbf{E}_\infty[\|z_0\|^2] : (1), (3)\}$$

Notice that the $\mathbf{E}_\infty[\|z_0\|^2]$ in the above performance can be computed as

$$\mathbf{E}_\infty[\|z_0\|^2] = \|T_{z_0 w_0}\|_2^2 + \sum_{j=1}^{n} \|T_{z_0 w_j} R_j^{1/2}\|_2^2 \zeta_j \tag{4}$$

if there exists a $\zeta = [\zeta_1 \quad \zeta_2 \quad \ldots \quad \zeta_n]^{\mathrm{T}}$ satisfying (3). The following theorem characterizes the conditions to bound this worst case performance J, which can be expressed by using the ϕ-measure defined before.

Theorem 2: *The worst-case performance J with respect to all FSN processes in $\mathcal{W}(\sigma, z)$ for a given noise-to-signal ratio vector σ satisfies $J < \gamma$ if and only if*

$$\phi\left(T_{\hat{z}\hat{w}}, \begin{bmatrix} \gamma^{-1} \\ \sigma \end{bmatrix}\right) < 1$$

Proof: See Appendix. □

Further analysis leads to the following computation for ϕ measure, which is useful in the sequential discussion.

Theorem 3: *For given* $\gamma \in \mathbb{R}_+$, $\sigma = [\sigma_1 \quad \sigma_2 \quad \ldots \quad \sigma_n]^{\mathrm{T}} \in \mathbb{R}_+^n$, *the ϕ measure for performance can be computed as*

$$\phi\left(T_{\hat{z}\hat{w}}, \begin{bmatrix} \gamma^{-1} \\ \sigma \end{bmatrix}\right) = \min_{e \in \mathbb{R}_+^{n+1}} \max_{0 \leq i \leq n, e_i \neq 0} \frac{\|T_{z_i \hat{w}} W_\gamma^{1/2}(e, \sigma)\|_2^2}{e_i}$$

where

$$W_\gamma(e, \sigma) \triangleq \text{block diag}\,(\gamma^{-1}\sigma_0 I, e_1\sigma_1 I, e_2\sigma_2 I, \ldots, e_n\sigma_n I)$$

For FSN or mean-square stability, the ϕ-measure is

$$\phi(T_{zw}, \sigma) = \phi\left(T_{\hat{z}\hat{w}}, \begin{bmatrix} 0 \\ \sigma \end{bmatrix}\right) = \min_{e \in \mathbb{R}_+^n} \max_{1 \leq i \leq n, e_i \neq 0} \frac{\|T_{z_i w} W^{1/2}(e, \sigma)\|_2^2}{e_i}$$

where

$$W(e, \sigma) \triangleq \text{block diag}\,(e_1\sigma_1 I, e_2\sigma_2 I, \ldots, e_n\sigma_n I)$$

Proof: Similar to those in Lu (1997) and Lu and Skelton (1997b). □

We have finished the analysis for both FSN stability and performance for systems with FSN models. The next task is to use these results to conduct integration of control and instrumentation.

4. Integrated instrumentation and control

Assume that each instrument variable (for example, actuator and sensor variables) corresponds to an instrument device. The plant P, including different I/O drive ports used for instruments, can be described by the following state space description

$$\left.\begin{aligned} \dot{x} &= Ax + \sum_{j=0}^{n} B_j w_j + B_u u \\ z_i &= C_i x + D_{iu} u \\ y &= C_y x + \sum_{j=0}^{n} D_{yj} w_j + D_{yu} u \end{aligned}\right\} \tag{5}$$

For feedback control purposes the following assumption for sysem (5) is used

Assumption 1: *(A, B_u) is a stabilizable pair and (A, C_y) is a detectable pair in the usual sense.*

The feedback control K sought here is of the following form

$$\left.\begin{aligned} \dot{x}_c &= A_c x + B_c y \\ u &= C_c x \end{aligned}\right\} \tag{6}$$

Denote the closed loop system from w_j to z_i as $T_{z_i w_i}(P, K)$, and from $\hat{w}$ to z_i as $T_{z_i \hat{w}}(P, K)$. The control system integration objectives studied here include: (i) choosing the instrumentation configuration; (ii) designing feedback control logic. That is, for a given performance requirement, select the hardware specifications for the sensors and actuators together with a feedback control algorithm. The instrumentation configuration is determined by the instrument finite-signal-to-noise effects, which are characterized by the FSN models. That is, the (w_i, z_i) pair for $i \in \{1, \ldots, n\}$ satisfies the relationships in Definition 1. The portion of the system without finite-signal-to-noise effect is called the *nominal system*, denoted as P_0, which has the following state space description

$$\left.\begin{aligned} \dot{x} &= Ax + B_0 w_0 + B_u u \\ z_i &= C_i x + D_{iu} u \\ y &= C_y x + D_{y0} w_0 + D_{yu} u \end{aligned}\right\} \tag{7}$$

Assume that the relative importance of the jth instrument in the total instrument cost is weighted by α_j; then the total instrument cost $\$$ can be expressed by

$$\$ = \sum_{j=1}^{n} \alpha_j \sigma_j^{-1}$$

The control system integration problem solves for the noise-to-signal ratio σ and the feedback controller K from the following optimization

$$\min_{\sigma, K} \left\{ \sum_{j=1}^{n} \alpha_j \sigma_j^{-1} : \max_{w \in \mathcal{W}(\sigma, z)} \mathbf{E}_\infty[\|z_0\|^2] \le \gamma \right\} \tag{8}$$

Notice that in order for (8) to have a solution the performance level γ must be achievable by the nominal plant P_0, i.e. there exists a controller such that, for the nominal plant P_0, we have

$$\mathbf{E}_\infty[\|z_0\|^2] \le \gamma$$

4.1. *FSN stability case*

Let us first consider the integrated design for FSN or mean-square stability, i.e. we want to find the instrumentation configuration with lowest instrument cost and a feedback controller K such that the control system is FSN stable. Mathematically, this can be expressed as

$$\min_{\sigma, K} \left\{ \sum_{j=1}^{n} \alpha_j \sigma_j^{-1} : \begin{array}{l} \text{such that the closed loop} \\ \text{system is FSN stable} \end{array} \right\}$$

Using Theorem 1, this problem can be expressed as

$$\min_{\sigma, K} \left\{ \sum_{j=1}^{n} \alpha_j \sigma_j^{-1} : \phi(T_{zw}(P, K), \sigma) < 1 \right\} \tag{9}$$

Since, if σ, K solve (9), there must exist a $\beta \ge 0$ such that σ, K solve the following unconstrained optimization

$$H = \min_{\sigma, K} \left\{ \sum_{j=1}^{n} \alpha_j \sigma_j^{-1} + \beta \phi(T_{zw}(P, K), \sigma) \right\} \tag{10}$$

therefore, in the following discussion, we study the optimization (10) for a given $\beta > 0$. In this optimization the noise-to-signal ratio σ and the feedback control law K depend on each other. As we know, there are many existing theories to synthesize feedback control laws. Constructing the noise-to-signal ratio σ as a function of the controller K might have certain advantages in the numerical computation. The following discussion is along this line of thinking.

Let

$$L(K) = \min_{\sigma} \left\{ \sum_{j=1}^{n} \alpha_j \sigma_j^{-1} + \beta \min_{e \in \mathbb{R}_+^n} \max_{1 \le i \le n, e_i \ne 0} \frac{\|T_{z_i w}(P, K) W^{1/2}(e, \sigma)\|_2^2}{e_i} \right\}$$

$$N(\sigma) = \sum_{j=1}^{n} \alpha_j \sigma_j^{-1} + \beta \min_{K, e} \max_{1 \le i \le n, e_i \ne 0} \frac{\|T_{z_i w}(P, K) W^{1/2}(e, \sigma)\|_2^2}{e_i}$$

then, by Theorem 3, we have

$$H = \min_K L(K) = \min_\sigma N(\sigma)$$

and it is also not hard to see that $N(\sigma)$ can be computed by the ϕ synthesis proposed in Lu (1997) and Lu and Skelton (1997b). An algorithm for the ϕ synthesis is the so-called $e-K$ iteration (a convergent algorithm), which solves the following problem

$$K = \arg\min_K \phi(T_{zw}(G,K),\sigma)$$

The discrete time version of this algorithm is reported in Lu and Skelton (1997b). The continuous time version can be found in Lu (1997). The following provides a summary of the continuous time version of this algorithm.

$e-K$ **iteration:**

(i) Set $k=0$ and choose an initial $e^k \in \mathbb{R}_+^{n+1}$.

(ii) Set $e = e^k$ and, for this e, solve the following multiobjective H_2 control problem

$$K_k = \arg\min_K \max_{1\le i\le n, e_i\neq 0} \frac{\|T_{z_i\omega}(G,K)W^{1/2}(e)\|_2^2}{e_i}$$

where

$$W(e) \triangleq \text{block diag}\,(e_1\sigma_1 I, e_2\sigma_2 I, \dots, e_n\sigma_n I).$$

(iii) Set $K = K_k$ and, for this K, solve the following eigenvalue problem

$$\phi_k = \max_{1\le i\le n, z, e_i\neq 0} \frac{\|T_{z_i w}(G,K)W^{1/2}(e)\|_2^2}{e_i}$$

with the eigenvector e^{k+1} satisfying $\sum_{i=1}^n e_i^{k+1} = 1$.

(iv) If the relative error at two sequential iterations satisfies

$$\frac{\|e^{k+1}-e^k\|}{\|e^k\|} + \frac{|\phi_{k+1}-\phi_k|}{\phi_k} < \epsilon$$

where ϵ is a given error tolerance, then stop. Otherwise, set $e^k \to e^{k+1}$ and $k = k+1$, go to step (ii).

This $e-K$ iteration converges to a locally optimal solution and the proof can be found in Lu (1997).

Now let us consider the computation for $L(K)$.

Theorem 4: *For a given controller K, the cost function $L(K)$ can be computed as*

$$L(K) = \min_{e\in\mathbb{R}_+^n} \max_{1\le i\le n, e_i\neq 0} 2\sqrt{\beta}\sum_{j=1}^n \sqrt{\frac{\alpha_j e_j}{e_i}}\|T_{z_i w_j}(P,K)\|_2$$

i.e.

$$H = \min_{K, e\in\mathbb{R}_+^n} \max_{1\le i\le n, e_i\neq 0} 2\sqrt{\beta}\sum_{j=1}^n \sqrt{\frac{\alpha_j e_j}{e_i}}\|T_{z_i w_j}(P,K)\|_2 \quad (11)$$

If K^*, e^*, i^* are the corresponding optimal values for the optimization (11)*, then the optimal noise-to-signal ratios for the optimization* (10) *can be computed as*

$$\sigma^* = \sqrt{\frac{e_{i^*}^*}{\beta}}\left[\sqrt{\frac{\alpha_1}{e_1^*}}\frac{1}{\|T_{z_{i^*}w_1(G,K^*)}\|_2}\sqrt{\frac{\alpha_2}{e_2^*}}\frac{1}{\|T_{z_{i^*}w_2(G,K^*)}\|_2}\cdots \times\sqrt{\frac{\alpha_n}{e_n^*}}\frac{1}{\|T_{z_{i^*}w_n(G,K^*)}\|_2}\right]^{\mathrm{T}} \quad (12)$$

Proof: See Appendix. □

Theorem 5: *$L(K)$ can be computed as the following*

$$L(K) = 2\sqrt{\beta}\bar{\lambda}(\mathtt{H}\sqrt{\text{diag}\,(\alpha)})$$

where the (i,j)th element of the matrix $\mathtt{H}$ *is* $[\mathtt{H}]_{ij} = \|T_{z_i w_j}(P,K)\|_2$, *and*

$$\alpha = [\alpha_1 \quad \alpha_2 \quad \dots \quad \alpha_n]^{\mathrm{T}}.$$

Let f be the eigenvector associated with the eigenvalue $\bar{\lambda}(\mathtt{H}\sqrt{\text{diag}\,(\alpha)})$ and satisfying

$$\sum_{j=1}^n f_j = 1$$

then the vector e which solves the optimization

$$L(K) = \min_{e\in\mathbb{R}_+^n} \min_{1\le i\le n, e_i\neq 0} 2\sqrt{\beta}\sum_{j=1}^n \sqrt{\frac{\alpha_j e_j}{e_i}}\|T_{z_i w_j}(P,K)\|_2$$

satisfies

$$e_j = f_j^2, \quad j\in\{1,\dots,n\}$$

Proof: See Appendix. □

Remarks 1: Equation (11) implies that, even for fixed vector e, the optimization is non-linear with respect to the feedback controller K. There is no existing method to solve (11) for this case. Instead of finding the globally optimal solution for (11), in the following an iterative algorithm is proposed to find a locally optimal solution. This iteration uses the results presented in Theorem 5, and is similar to the so-called $e-K$ iteration proposed in Lu (1997) and Lu and Skelton (1997b), except for the adjustment of the optimal noise-to-signal ratios at each iteration. Because of this fact, we should call this algorithm $\sigma-K$ iteration in comparison with $e-K$ iteration.

$\sigma-K$ **iteration:**

Step 1. Set $k=0$. Given the noise-to-signal ratio vector σ, find a feedback controller K (using $e-K$ iteration) to solve

$$\phi_k = \min_{K,e} \max_{1\le i\le n, e_i\neq 0} \frac{\|T_{z_i\hat{w}}(P,K)W_\gamma^{1/2}(e,\sigma)\|_2^2}{e_i}$$

Step 2. For the controller K obtained in step 1, find the eigenvector f associated with the eigenvalue $\bar{\lambda}(\mathrm{H}\sqrt{\mathrm{diag}\,(\alpha)})$ such that

$$\sum_{j=1}^{n} f_j = 1$$

and find the index i such that

$$\bar{\lambda}(\mathrm{H}\sqrt{\mathrm{diag}\,(\alpha)}) = \sum_{j=1}^{n} \frac{\sqrt{\alpha_j} f_j}{f_i} \| T_{z_i w_j}(P,K)\|_2$$

Step 3. Compute with respect to the vector f and the index i found in Step 2

$$\tilde{\sigma} = \frac{f_i}{\sqrt{\beta}} \left[\frac{\sqrt{\alpha_1}}{f_1} \frac{1}{\|T_{z_i w_1}(P,K)\|_2} \frac{\sqrt{\alpha_2}}{f_2} \frac{1}{\|T_{z_i w_2}(P,K)\|_2} \cdots \times \frac{\sqrt{\alpha_n}}{f_n} \frac{1}{\|T_{z_i w_n}(P,K)\|_2} \right]^{\mathrm{T}}$$

Step 4. If $\|\sigma - \tilde{\sigma}\| \le \epsilon$, stop; otherwise set $\sigma = \tilde{\sigma}$, $k = k+1$ and go to Step 1.

Theorem 6: *The above* $\sigma - K$ *iteration is a convergent algorithm.*

Proof: See Appendix. □

Designing the control law and the instrumentation configuration to achieve mean-square stability is not of much practical significance due to the fact that a system maintaining mean-square stability could have very bad performance. However, the discussion here shows the difficulty of solving the integrated instrumentation and control problem. In the next subsection, we study the integrated design for performance.

4.2. *Performance case*

Using Theorem 2, the integrated design problem for mean-square performance defined in (8) can be equivalently expressed as

$$\min_{\sigma} \sum_{j=1}^{n} \alpha_j \sigma_j^{-1}$$

subject to

$$\exists K \text{ such that } \phi\left(T_{\hat{z}\hat{w}}(P,K), \begin{bmatrix} \gamma^{-1} \\ \sigma \end{bmatrix} \right) \le 1$$

or further expressed as

$$\min_{\sigma} \sum_{j=1}^{n} \alpha_j \sigma_j^{-1}$$

subject to

$$\min_{K} \phi\left(T_{\hat{z}\hat{w}}(P,K), \begin{bmatrix} \gamma^{-1} \\ \sigma \end{bmatrix} \right) \le 1 \quad (13)$$

where

$$\phi\left(T_{\hat{z}\hat{w}}(P,K), \begin{bmatrix} \gamma^{-1} \\ \sigma \end{bmatrix} \right) = \min_{e} \max_{i} \left\{ \frac{e_0}{e_i \gamma} \|T_{z_i w_0}(P,K)\|_2^2 + \sum_{j=1}^{n} \frac{e_j}{e_i} \|T_{z_i w_j}(P,K)\|_2^2 \sigma_j \right\}$$

Notice that, for fixed γ,

$$\phi^{-1}\left(T_{\hat{z}\hat{w}}(P,K), \begin{bmatrix} \gamma^{-1} \\ \sigma \end{bmatrix} \right)$$

is proportional to the size of the set R^{σ} or the noise-to-signal ratio σ. Hence in (13) the minimization of the ϕ-measure *indirectly* contributes to reducing the instrumentation cost

$$\$ = \sum_{j=1}^{n} \alpha_j \sigma_j^{-1}$$

Assume an optimal controller K, the vector e and the optimal index i for the following optimization

$$\min_{K} \phi\left(T_{\hat{z}\hat{w}}(P,K), \begin{bmatrix} \gamma^{-1} \\ \sigma \end{bmatrix} \right)$$

are given, then

$$\phi\left(T_{\hat{z}\hat{w}}(P,K), \begin{bmatrix} \gamma^{-1} \\ \sigma \end{bmatrix} \right) = \frac{e_0}{e_i \gamma} \|T_{z_i w_0}(P,K)\|_2^2 + \sum_{j=1}^{n} \frac{e_j}{e_i} \|T_{z_i w_j}(P,K)\|_2^2 \sigma_j$$

In this case, (13) can be simplified into the following optimization

$$\min_{\sigma} \sum_{j=1}^{n} \alpha_j \sigma_j^{-1}$$

subject to

$$\frac{e_0}{e_i \gamma} \|T_{z_i w_0}(P,K)\|_2^2 + \sum_{j=1}^{n} \frac{e_j}{e_i} \|T_{z_i w_j}(P,K)\|_2^2 \sigma_j \le 1 \quad (14)$$

Remark 2: Notice that a necessary condition for the constraint in (14) to be feasible is that

$$\|T_{z_0 w_0}(P,K)\|_2^2 \le \gamma$$

Hence, in order to conduct the integrated design, all the conditions which guarantee the existence of a controller for the nominal plant P_0 need to be carried out here. This is the reason for Assumption 1.

Using a Khun–Tucker multiplier, the optimal solution for (14) occurs at

$$\sigma = \sqrt{\frac{e_i}{\beta}}\left[\sqrt{\frac{\alpha_1}{e_1}}\frac{1}{\|T_{z_iw_1}(P,K)\|_2}, \sqrt{\frac{\alpha_2}{e_2}}\frac{1}{\|T_{z_iw_2}(P,K)\|_2}\cdots\sqrt{\frac{\alpha_n}{e_n}}\frac{1}{\|T_{z_iw_1}(P,K)\|_2}\right]^{\mathrm{T}}$$

where β is the multiplier and can be computed as

$$\sqrt{\beta} = \frac{\sum_{j=1}^{n}\sqrt{\frac{e_j\alpha_j}{e_i}}\|T_{z_iw_j}(P,K)\|_2}{1-\frac{e_0}{e_i\gamma}\|T_{z_iw_0}(P,K)\|_2^2} \tag{15}$$

and the optimal instrument cost is

$$\$ = \sum_{j=1}^{n}\alpha_j\sigma_j^{-1} = \frac{\left\{\sum_{j=1}^{n}\sqrt{\frac{\alpha_je_j}{e_i}}\|T_{z_iw_j}(P,K)\|_2\right\}^2}{1-\frac{e_0}{e_i\gamma}\|T_{z_iw_0}(P,K)\|_2^2} \tag{16}$$

Equation (16) implies that the instrumentation cost is also proportional to the gap between the nominal closed loop system performance and the performance bound γ. If the gap is large, there is room to choose coarse instruments, but if the gap is small more precise instruments must be chosen for a given performance requirement.

In the design situation, K in (14) must be designed. For the above constrained optimization, the first desire is to make the constraint feasible, i.e.

$$\min_K \phi\left(T_{\hat{z}\hat{w}}(P,K), \begin{bmatrix}\gamma^{-1}\\ \sigma\end{bmatrix}\right) \le 1$$

The following provides a method to adjust both the noise-to-signal ratio σ and the controller K such that this constraint is kept feasible.

Theorem 7: *Consider the following iteration for ϕ_k and σ^k with a given γ and initial noise-to-signal ratio σ^0*

$$\left.\begin{aligned}\sigma^k &= \sigma^{k-1}/\phi_{k-1}\\ \phi_{k+1} &= \min_K \phi\left(T_{\hat{z}\hat{w}}(P,K), \begin{bmatrix}\gamma^{-1}\\ \sigma^k\end{bmatrix}\right),\\ &\qquad k\in\{1,2,\ldots,\infty\}\end{aligned}\right\} \tag{17}$$

where

$$\phi_0 = \min_K \phi\left(T_{\hat{z}\hat{w}}(P,K), \begin{bmatrix}\gamma^{-1}\\ \sigma^0\end{bmatrix}\right)$$

If $\phi_0 > 1$, then this iteration converges. Furthermore, there exists $\sigma^* \in \mathbb{R}_+^n$ such that

$$\lim_{k\to\infty}\sigma^k = \sigma^*, \quad \lim_{k\to\infty}\phi_k = 1$$

Proof: See Appendix. □

Similarly to the FSN stability case, the problem (13) cannot be solved exactly by existing algorithms (i.e. cannot be solved exactly by computationally tractable algorithms). In the following, we consider an iterative algorithm to find a locally optimal solution for (13). This algorithm iterates between solving problem (14) for an optimal noise-to-signal ratio σ and solving K, σ to enforce the constraint

$$\min_K \phi\left(T_{\hat{z}\hat{w}}(P,K), \begin{bmatrix}\gamma^{-1}\\ \sigma\end{bmatrix}\right) = 1$$

The latter can be achieved by using Theorem 7. Because of the feature of enforcing the ϕ measure and adjusting the noise-to-signal ratio σ, we should call this algorithm the $\sigma - \phi$ iteration.

$\sigma - \phi$ **iteration:**

Step 1. Set $k = 0$. Given a vector σ, find a feedback controller K and a vector e to solve

$$\sigma_k = \min_{K,e\in\mathbb{R}^{n+1}}\ \max_{0\le i\le n, e_i\ne 0}\frac{\|T_{z_i\hat{w}}(P,K)W_\gamma^{1/2}(e,\sigma)\|_2^2}{e_i}$$

using the $e - K$ iteration. If $\phi_k \not< 1$, conduct iteration in (17) until $\phi_k \le 1+\epsilon$ and set σ to be this σ^k.

Step 2. For the given K and e, find the index i such that

$$\phi_k = \max_i\left\{\frac{e_0}{e_i\gamma}\|T_{z_iw_0}(P,K)\|_2^2 + \sum_{j=1}^{n}\frac{e_j}{e_i}\|T_{z_iw_j}(P,K)\|_2^2\sigma_j\right\}$$

and compute the noise-to-signal ratio vector from (15); denote it as $\tilde{\sigma}$.

Step 3. If $\|\sigma - \tilde{\sigma}\| \le \epsilon$, stop; otherwise set $\sigma = \tilde{\sigma}$, $k = k+1$ and go to Step 1.

Remark 3: Notice that $\phi_k \le 1$ guarantees

$$\frac{e_0}{e_i\gamma}\|T_{z_iw_0}(P,K)\|_2^2 < 1$$

hence the noise-to-signal ratios computed in Step 2 are positive numbers.

Remark 4: If the above $\sigma - \phi$ iteration converges, a locally optimal solution for the controller K and the noise-to-signal ratio σ can be achieved, due to the fact that Step 2 minimizes the instrument cost. However, unlike the $e - K$ iteration and the $\sigma - K$ iteration, the convergence of this algorithm cannot be guaranteed.

5. Example

A typical control system can be depicted as in figure 1, where $A\hat{P}S$ is the augmented plant P whose state space description is

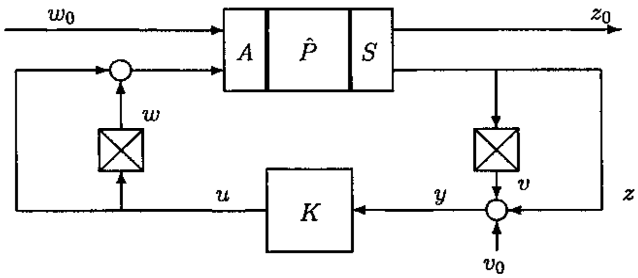

Figure 1. A controlled system with sensor and actuator finite-signal-to-noise model.

$$
\left.\begin{aligned}
\dot{x} &= Ax + B_0 w_0 + \sum_{j=1}^{n_u} B_j(u_j + w_j) \\
z_0 &= C_0 x + \sum_{j=1}^{n_u} D_{0_j} u_j \\
z_i &= C_i x + D_{y_0} w_0 + \sum_{j=1}^{N} D_{y_j}(u_j + w_j) \\
y_i &= z_i + v_i + v_{0_i}, \quad i \in \{1, 2, \ldots, n_y\}
\end{aligned}\right\} \qquad (18)
$$

where S and A denote the sensor and the actuator devices, and $\hat{P}$ is the plant to be controlled. K in figure 1 represents a feedback controller. w_0 and v_{0_i} for $i \in \{1, 2, \ldots, n_y\}$ denote the external white noise process defined in Definition 1. Consider the sensors and the actuators having finite-signal-to-noise effects. These effects are characterized by the noises w and v, depicted by the '×' boxes in figure 1.

Mathematically, the finite-signal-to-noise effect of the actuators implies the following FSN model

$$W_j = R_j \mathbf{E}[\|u_j\|^2], \quad 0 \le R_j \le \sigma_j I$$

where σ_j^{-1} for $j \in \{1, 2, \ldots, n_u\}$ are the signal-to-noise ratios associated with the actuator instruments, and W_j is the spectral density of w_j. Similarly, the finite-signal-to-noise effect of sensor instruments can be characterized by

$$V_j = R_j \mathbf{E}[\|z_j\|^2], \quad 0 \le R_j \le \sigma_{n_u+j} I$$

for $j \in \{1, 2, \ldots, n_y\}$, where $\sigma_{n_u+j}^{-1}$ is the jth signal-to-noise ratio associated with the jth sensor instrument and V_j is the spectral density of v_j. The integrated instrumentation and control problem considered here finds the noise-to-signal ratios associated with the sensors and actuators and a feedback control law such that the closed loop system performance level can be achieved by instrumentation configuration with as low cost as possible.

More specifically, let us consider the control system integration for civil structure applications. In the face of current pressing problems in the deteriorating civil infrastructure worldwide, and to enhance structural functionality and safety against natural hazards such as earthquakes and high wind, great interest has been generated all around the world in the application of active controls. The civil structure application requires a reliable and cost-effective means of reducing the response of the structure to environmental loads. The key issues in civil structure control include large power consumption and high instrument cost.

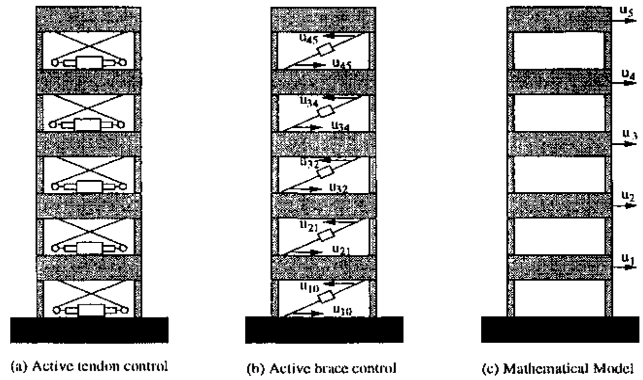

Figure 2. A five-storey building with active control.

In this section we will use the $\sigma - \phi$ algorithm proposed in the previous section to design a control system with low instrument cost in the lateral vibration control of a five-storey building, depicted in figure 2. Figure 2c represents our simplified model to illustrate the main ideas of this paper; a more realistic model will be studied later. This building is subjected to a one-dimensional earthquake excitation $\ddot{x}_g$ at its base and another one-dimensional force excitation at the top (to simulate the wind disturbance applied to the building). The natural damping and stiffness might not be enough for the building to suppress adequately the vibrations caused by these disturbances. Hence the active controls are applied; however, the control hardware must have very low cost.

Denote m_i, d_i, k_i as the mass, damping and stiffness coefficients of the ith floor of the building; then the equation of motion of the building can be expressed as

$$M(\ddot{q} + B_g \ddot{x}_g) + D\dot{q} + Eq = B_2 u + B_t w_t$$

where $q(t) = [q_5(t) q_4(t) \ldots q_1(t)]^T$ denotes the floor displacements relative to the ground, and

$$M = \mathrm{diag}(m_5, m_4, \ldots, m_1)$$

$$D = \begin{bmatrix} d_5 & -d_5 & & & \\ -d_5 & d_5 + d_4 & -d_4 & & \\ & -d_4 & d_4 + d_3 & -d_3 & \\ & & -d_3 & d_3 + d_2 & -d_2 \\ & & & -d_2 & d_2 + d_1 \end{bmatrix}$$

$$E = \begin{bmatrix} k_5 & -k_5 & & & \\ -k_5 & k_5+k_4 & -k_4 & & \\ & -k_4 & k_4+k_3 & -k_3 & \\ & & -k_3 & k_3+k_2 & -k_2 \\ & & & -k_2 & k_2+k_1 \end{bmatrix}$$

$\ddot{x}_g$ and w_t denote the base acceleration disturbance and the top wind disturbance,

$$B_g = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad B_t = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad B_2 = I$$

and we take the spectral densities of $\ddot{x}_q$ and w_t as

$$S_{\ddot{x}_g} = 1, \quad S_{w_t} = 0.05$$

In order to control the lateral vibration of the building, we measure the absolute floor accelerations. The sensor output can be expressed as

$$y(t) = \ddot{q} + B_b\ddot{x}_g + v_0 + v$$

The spectral densities of the external sensor noises are taken as $S_{v_{0_i}} = 0.01$, $i \in \{1, 2, \dots, n_y\}$. The control variables are $u_1, u_2, \dots, u_5$. The normalized parameters are summarized in table 1.

The instrumentation cost is defined as

$$\text{Instrument cost } \$ = \sum_{i=1}^{n_u+n_y} \alpha_i \sigma_i^{-1}$$

the weights are taken as

$$\alpha = [1 \quad 1 \quad \dots \quad 1]^{\mathrm{T}}$$

i.e. we assume that all the instruments have the same price.

We consider two control objectives:

(Objective 1) Design a control and the signal-to-noise ratios associated with the sensors and actuators to achieve the variance bound for the interstorey drift

$$z_{\mathrm{isd}}(t) = \begin{bmatrix} 1 & -1 & & & \\ & 1 & -1 & & \\ & & 1 & -1 & \\ & & & 1 & -1 \end{bmatrix} q(t)$$

m_1	m_2	m_3	m_4	m_5
1	0.9	0.8	0.7	0.6
k_1	k_2	k_3	k_4	k_5
1	0.9	0.8	0.7	0.6
d_1	d_2	d_3	d_4	d_5
0.020	0.018	0.016	0.014	0.012

Table 1. Summary of the system parameters.

(Objective 2) Design a control and the signal-to-noise ratios associated with the sensors and actuators to achieve the variance bound for the absolute floor accelerations

$$z_a(t) = \ddot{q}(t) + B_g\ddot{x}_g$$

Let us consider an initial control configuration with a given signal-to-noise ratio vector

$$\sigma = 0.5[1 \quad 1 \quad \dots \quad 1] \qquad (19)$$

For this instrumentation configuration, the instrument cost is

$$\text{Instrument cost } \$ = 20$$

We want to design a feedback control law together with the instrument noise-to-signal ratios to optimize the instrumentation cost for some performance requirements.

Let us first design controller Ks for the initial instrumentation configuration (19) for objective 1 and objective 2. The achievable optimal performances for fixed noise-to-signal ratio σs in (19) will be used as base performances for integrated design (simultaneously designing K and σ). For control objective 1, we conduct $e-K$ iteration and a line search for γ to find a controller such that

$$\max_{\text{FSN effect (19)}} \mathbf{E}_\infty[\|z_{\mathrm{isd}}\|^2] \le \gamma_1^{\mathrm{base}}$$

The $e-K$ iterations for different performance bounds are shown in figure 3, where each curve terminates at an iteration index which is the total number of iterations needed for the $e-K$ iteration to converge with an error tolerance $\epsilon = 10^{-3}$. The relationship between the final ϕ value obtained in $e-K$ iteration and the different performance bound γs (line search for γ) is shown in

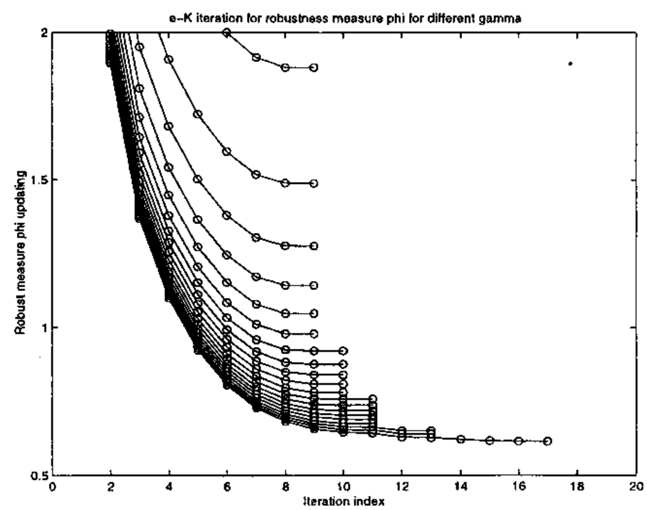

Figure 3. $e-K$ iterations for control objective 1 with instrumentation configuration (19): the eigenvalues converge to ϕ-measures for different performance requirements. Lower ϕ-curves correspond to smaller γs.

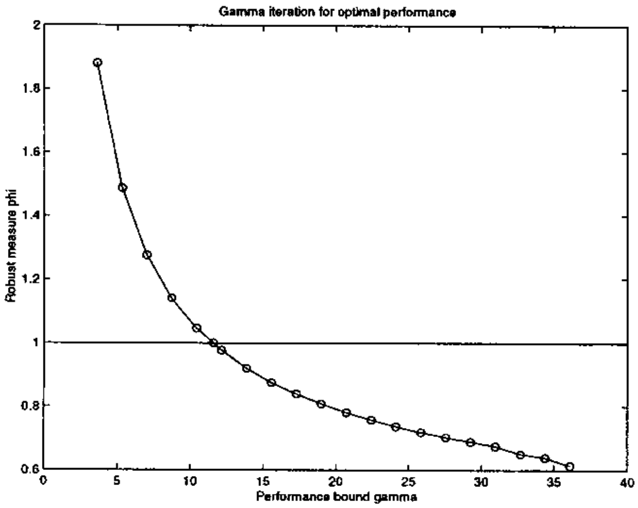

Figure 4. The final ϕ measure versus different γs for control objective 1 with instrumentation configuration (19).

figure 4. The optimal performance level is the γ value corresponding to $\phi = 1$. From figure 4, this optimal performance is

$$\text{Objective } 1\, \gamma_1^{\text{base}} = 11.5925$$

For control objective 2, similarly, the optimal performance level for instrumentation configuration (19) can be obtained as

$$\text{Objective } 2\, \gamma_2^{\text{base}} = 4.3822$$

Now we are ready to redesign both the control logic K and the noise-to-signal ratios based on the configuration (19). The goal of the integrated instrumentation and control is to find the noise-to-signal ratios and a control law K such that the worst-case output variances of $z_{\text{isd}}(t)$ or $z_{\text{a}}(t)$ are bounded by γ_1 and γ_2, i.e.

$$\max_{\text{FSN models}} \mathbf{E}_\infty[\|z_{\text{isd}}\|^2] \le \gamma_1, \max_{\text{FSN models}} \mathbf{E}_\infty[\|z_a\|^2] \le \gamma_2$$

The performance bounds here are chosen as the following, based on the optimal performances for the instrumentation configuration (19)

$$\gamma_i = [0.5 \quad 1 \quad 1.5]\gamma_i^{\text{base}}, \quad i = 1, 2$$

where $\gamma_i = 0.5\gamma_i^{\text{base}}$ corresponds to the most stringent performance requested. With the instrumentation configuration as in (19), there is no control to achieve this performance level. $\gamma_i = 1.5\gamma_i^{\text{base}}$ corresponds to lenient performance, and there exists a feedback controller for the instrumentation configuration (19) to achieve this performance. We use the $\sigma - \phi$ iteration to conduct the integrated design.

5.1. *Numerical study for control objective 1*

For the stringent performance requirement $\gamma_1 = 0.5\gamma_1^{\text{base}}$, the $\sigma - \phi$ iteration converges after the 14th iteration for an error tolerance $\epsilon = 0.005$. The instrument costs corresponding to iteration numbers are shown in figure 5. Shown in figure 6 are the distributions of the variances of the control energy, the displacements relative to the ground, the interstorey drifts and the floor accelerations, and the instrument signal-to-

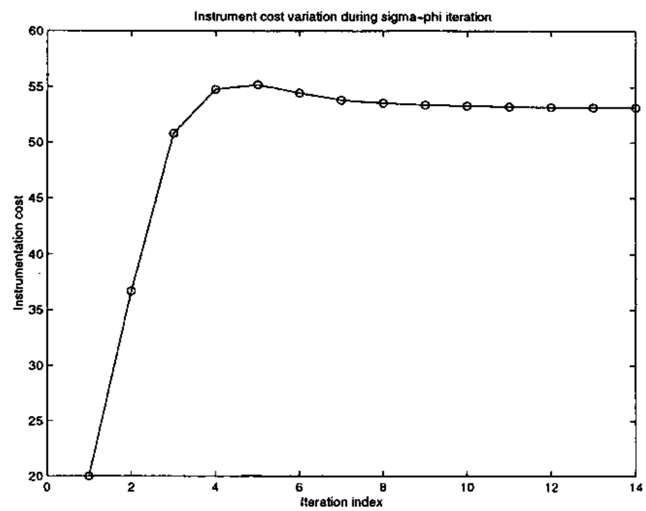

Figure 5. For control objective 1 with stringent performance requirement $\gamma_1 = 0.5\gamma_1^{\text{base}}$, the instrumentation cost during the $\sigma - \phi$ iteration.

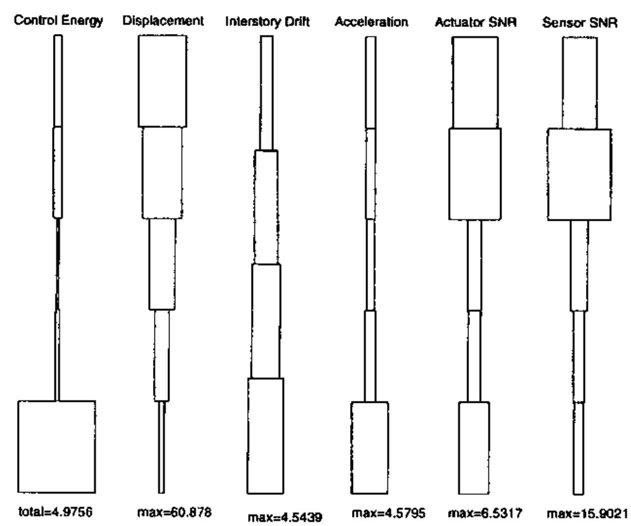

Figure 6. For control objective 1 with stringent performance requirement $\gamma_1 = 0.5\gamma_1^{\text{base}}$, the distributions of the variances of the control energy, the displacements relative to the ground, the interstorey drifts and the floor accelerations, and the instrument signal-to-noise ratios along the floors.

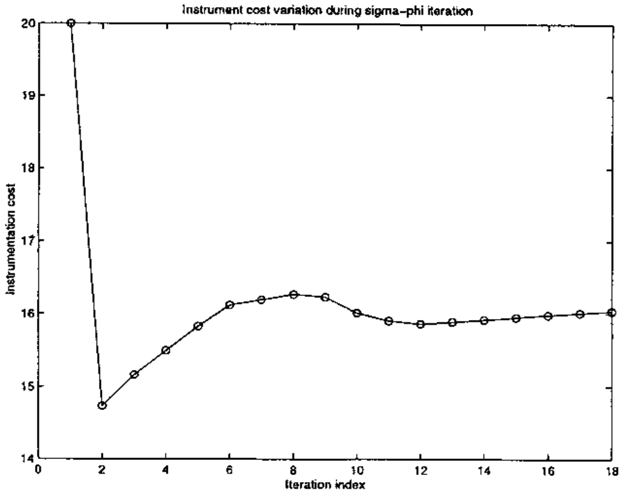

Figure 7. For control objective 1 with performance $\gamma_1 = \gamma_1^{\text{base}}$, the instrumentation cost during the $\sigma - \phi$ iteration.

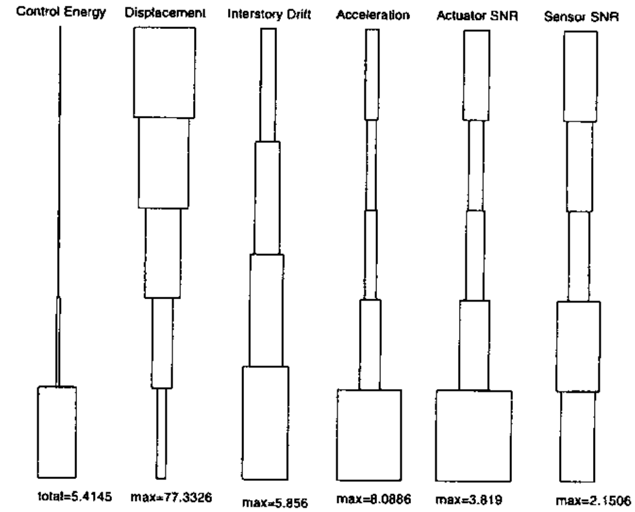

Figure 8. For control objective 1 with performance $\gamma_1 = \gamma_1^{\text{base}}$, the distributions of the variances of the control energy, the displacements relative to the ground, the interstorey drifts and the floor accelerations, and the instrument signal-to-noise ratios along the floors.

noise ratios along the floors. We can see from figure 6 that the finest actuator and sensor instruments are placed on the fourth floor. Sensors on the first, second and third floors could be very coarse, while the actuator devices on the second and third floors could be very coarse.

For the performance requirement which matches that for the instrumentation configuration (19), the $\sigma - \phi$ iteration converges after the 18th iteration for an error tolerance $\epsilon = 0.005$. The instrument cost decreases as compared to the configuration (19). The instrument costs corresponding to iteration numbers are shown in figure 7, while the various distributions along the floors are shown in figure 8. For this performance requirement, the finest sensor is on the second floor whereas the finest actuator is on the first floor, which is different from the stringent performance case. Hence we conclude that the instrumentation configuration depends on the performance requirement. Notice that the control energy increases over the stringent performance case, indicating that cheaper instrumentation might require more control energy.

For lenient performance level, the $\sigma - \phi$ converges after the 24th iteration. The instrument cost also decreases in comparison with the instrumentation configuration (19); see figures 9 and 10 for details. For this performance requirement, the finest sensors are placed on the bottom two floors, while the finest actuator is placed on the bottom floor.

Table 2 summarizes the results for the integrated instrumentation and control design for control objective 1. We find that the control energy is not a monotonic function of the performance.

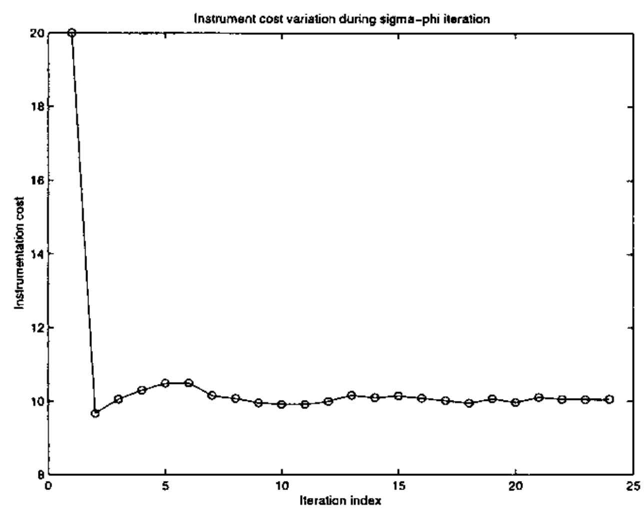

Figure 9. For control objective 1 with performance $\gamma_1 = 1.5\,\gamma_1^{\text{base}}$, the instrumentation cost during the $\sigma - \phi$ iteration.

Remark 5: Notice that, without FSN models, the use of the standard LQG theory would require infinite control energy, since our performance does not penalize the control variables. However, the FSN control design yields maximal accuracy (minimum variance) at finite control energy. This is due to the fact that noise increases with control energy in the FSN models. The significance of this fact for robust control should be investigated.

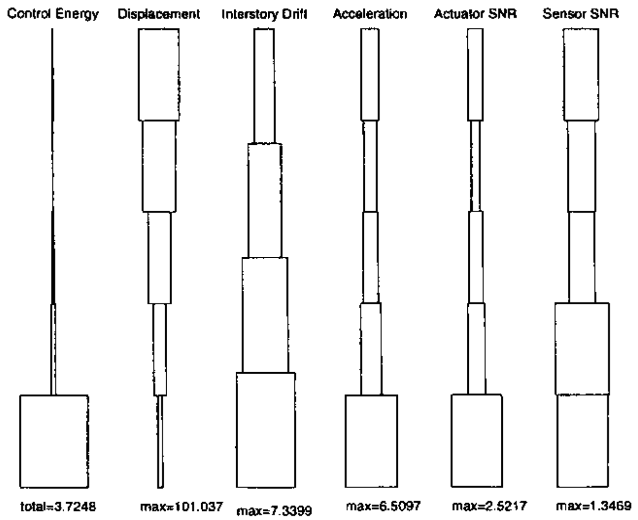

Figure 10. For control objective 1 with performance $\gamma_1 = 1.5\,\gamma_1^{\text{base}}$, the distributions of the variances of the control energy, the displacements relative to the ground, the interstorey drifts and the floor accelerations, and the instrument signal-to-noise ratios along the floors.

	$0.5\,\gamma_1^{\text{base}}$	γ_1^{base}	$1.5\,\gamma_1^{\text{base}}$
Instrumentation cost	53.1388	16.0383	10.0504
Control energy	4.9756	5.4145	3.7248
Sensor SNRs	$\begin{bmatrix} 8.5524 \\ 15.9021 \\ 4.2376 \\ 2.8366 \\ 2.3396 \end{bmatrix}$	$\begin{bmatrix} 1.6500 \\ 1.2848 \\ 1.0236 \\ 2.1506 \\ 1.7043 \end{bmatrix}$	$\begin{bmatrix} 0.8599 \\ 0.6796 \\ 0.6453 \\ 1.3469 \\ 1.2605 \end{bmatrix}$
Actuator SNRs	$\begin{bmatrix} 5.6634 \\ 6.5317 \\ 1.7072 \\ 1.6196 \\ 3.7487 \end{bmatrix}$	$\begin{bmatrix} 1.2732 \\ 0.7300 \\ 0.9182 \\ 1.4845 \\ 3.8190 \end{bmatrix}$	$\begin{bmatrix} 0.7568 \\ 0.4317 \\ 0.6951 \\ 0.8574 \\ 2.5217 \end{bmatrix}$

Table 2. Summary of the integrated design for control objective 1.

5.2. *Numerical study for control objective 2*

For the stringent performance requirement $\gamma_2 = 0.5\gamma_2^{\text{base}}$, the $\sigma - \phi$ iteration does not converge. For the performance requirement $\gamma_2 = \gamma_2^{\text{base}}$, the $\sigma - \phi$ converges after the 32nd iteration and the instrument cost converges to 20.3224. The instrument costs corresponding to iteration numbers are shown in figure 11. Shown in figure 12 are the distributions of the variances of the control energy, the displacements relative to the ground, the interstorey drifts and the floor accelerations,

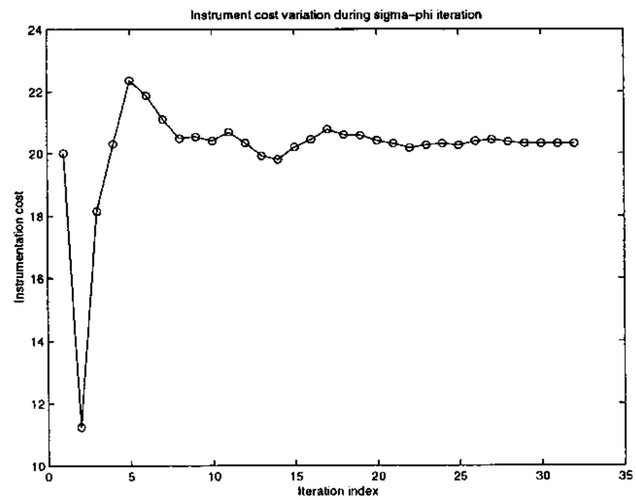

Figure 11. For control objective 2 with performance $\gamma_2 = \gamma_2^{\text{base}}$, the instrumentation cost during the $\sigma - \phi$ iteration.

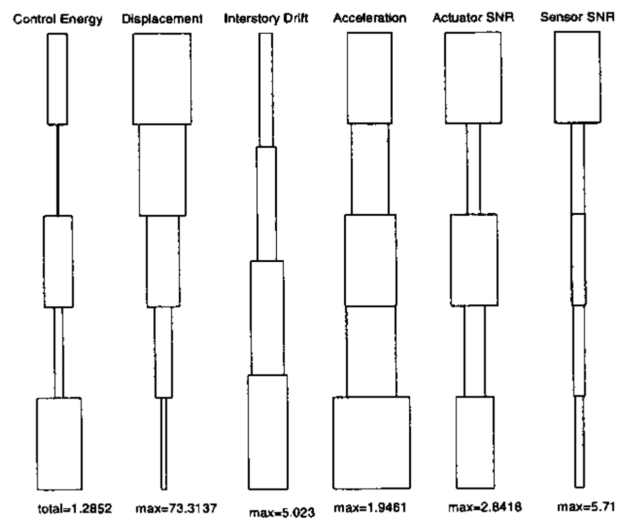

Figure 12. For control objective 2 with performance $\gamma_2 = \gamma_2^{\text{base}}$, the distributions of the variances of the control energy, the displacements relative to the ground, the interstorey drifts and the floor accelerations, and the instrument signal-to-noise ratios along the floors.

and the instrument signal-to-noise ratios along the floors. We can see from figure 12 that the finest sensor instrument is placed on the top floor. Three fine actuators are placed on the first, third and fifth floors. The control energies are also mainly distributed on the first, third and fifth floors. The floor accelerations are much smaller than those obtained in control objective 1. Also, this design has small interstorey drift and smaller control energy than those obtained in control objective 1. Hence we conclude that the control objective 2 is a better choice for control design.

6. Conclusion

Instrument precision is characterized by its signal-to-noise ratio, employing a new noise model. In this paper, this measure of precision of each instrument (an instrument is any finite precision component of a system) is optimized jointly with the control design, so as to use the least precision in each instrument, subject to an L_2 performance requirement for the closed loop system. This paper provides a first attempt to conduct both theoretical and numerical study for this joint optimization.

If the financial cost of a component is proportional to its signal-to-noise ratio, then the problem studied here can be interpreted as an 'economic' design in the sense that when the optimized precision of a component is small, the component does not need high precision and may be purchased from off-the-shelf hardware, or perhaps the component can be deleted altogether without a large impact on the system performance. Therefore the solution obtained here identifies where to spend money on a system, beginning from closed loop performance criteria and generating design requirements for each component of the system. The solution obtained here can also be used to identify the performance-limiting components of a closed loop system.

It is not hard to see the following from the example studied in this paper: the instrumentation configurations are affected not only by control objectives but also by the required performance level; there is no necessary relationship between the distribution of the signal-to-noise ratios and the distribution of the control energies, i.e. coarse instruments may or may not need to consume a large portion of the control energy in order to fulfil the control goal.

Notice that this paper presents a small step towards a rather broad integration of control and instrumentation design. Future work will focus on other integrations.

Acknowledgment

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Appendix: Proofs of theorems

Proof of Theorem 1: $\forall \zeta, \hat{\zeta} \in \mathbb{R}^n_+$, define $\tilde{\zeta} = \zeta - \hat{\zeta}$. Consider the difference

$$\mathcal{V}(\zeta) - \mathcal{V}(\hat{\zeta}) = \mathtt{G}_R \tilde{\zeta}$$

where $R = \text{block diag}\,(R_1, R_2, \ldots, R_n) \in \boldsymbol{R}^\sigma$.

(ii) implies that for all $R \in \boldsymbol{R}^\sigma$

$$\|\mathtt{G}_R \tilde{\zeta}\| \le \|\tilde{\zeta}\|$$

or there exists an induced matrix norm $\|\cdot\|_{\text{in}}$ associated with the vector norm $\|\cdot\|$ such that

$$\max_{R\in\boldsymbol{R}^\sigma} \|\mathtt{G}_R\|_{\text{in}} < 1$$

Since the matrix spectral radius is less than any induced matrix norm, the above is equivalent to

$$\max_{R\in\boldsymbol{R}^\sigma} \rho(\mathtt{G}_R) < 1$$

Consider that $\mathtt{G}_R$ is a matrix of all positive elements; by the Perron–Frobenius theorem $\rho(\mathtt{G}_R) = \bar{\lambda}(\mathtt{G}_R)$. Therefore (ii) $\Leftrightarrow$ (iii).

(ii) $\Rightarrow$ (i) is a standard result, hence (iii) $\Rightarrow$ (i). Now we assume that (iii) fails, i.e. there exists a $R = \text{block diag}\,(R_1, R_2, \ldots, R_n)$ such that

$$\bar{\lambda}(\mathtt{G}_R) \not< 1$$

Since $\bar{\lambda}(\mathtt{G}_R)$ is a continuous function of R and $\mathtt{G}_0 = 0$, there must exist $\hat{R} \in \boldsymbol{R}^\sigma$ such that

$$\bar{\lambda}(\mathtt{G}_{\hat{R}}) = 1$$

or $I - \mathtt{G}_{\hat{R}}$ is singular. This implies that there is no unique solution for ζ satisfying $\zeta = \mathcal{V}(\zeta)$. Therefore, if (ii) fails, $\mathcal{V}(\cdot)$ does not have a unique solution for ζ, i.e. (i) fails.

Hence we proved that (ii) $\Leftrightarrow$ (iii) $\Rightarrow$ (i) and not (iii) $\Rightarrow$ not (i), i.e. (ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (i).

Now, we want to prove

$$\max_{R\in\boldsymbol{R}^\sigma} \bar{\lambda}(\mathtt{G}_R) = \bar{\lambda}(\mathtt{H}\,\text{diag}\,(\sigma))$$

Denote the ith row of $\mathtt{G}_R$ as $[\mathtt{G}_R]_{i*}$. Using the Perron–Frobenius theorem, we have the following equivalence

$$\begin{aligned}
\max_{R\in\boldsymbol{R}^\sigma} \bar{\lambda}(\mathtt{G}_R) &= \max_{R\in\boldsymbol{R}^\sigma} \max_{e\in\mathbb{R}^n_+} \min_i \left\{ [\mathtt{G}_R]_{i*} \frac{e}{e_i} \right\} \\
&= \max_{e\in\mathbb{R}^n_+} \max_{R\in\boldsymbol{R}^\sigma} \min_i \left\{ [\mathtt{G}_R]_{i*} \frac{e}{e_i} \right\} \\
&\le \max_{e\in\mathbb{R}^n_+} \min_i \max_{R\in\boldsymbol{R}^\sigma} \left\{ [\mathtt{G}_R]_{i*} \frac{e}{e_i} \right\} \\
&= \max_{e\in\mathbb{R}^n_+} \min_i \left\{ [\mathtt{G}\,\text{diag}\,(\sigma)]_{i*} \frac{e}{e_i} \right\} \\
&= \bar{\lambda}(\mathtt{H}\,\text{diag}\,(\sigma)).
\end{aligned}$$

Note that the fourth equality comes from the fact that

$$\|T_{z_i w_j} R_j^{1/2}\|_2 \le \|T_{z_i w_j}\|_2 \sigma_j, \quad \forall R_j \le \sigma_j I$$

On the other hand, block diag $(\sigma_1 I, \sigma_2 I, \ldots, \sigma_n I) \in \boldsymbol{R}^\sigma$ implies

$$\bar{\lambda}(\mathtt{G}\,\text{diag}\,(\sigma)) \le \max_{R\in\boldsymbol{R}^\sigma} \bar{\lambda}(\mathtt{G}_R)$$

hence we must have

$$\max_{R\in\boldsymbol{R}^\sigma} \bar{\lambda}(\mathtt{G}_R) = \bar{\lambda}(\mathtt{G}\,\text{diag}\,(\sigma))$$

Therefore (ii) is equivalent to (iii).

Let (iv) be true i.e., there exists $X \ge 0$ and $Z_j \ge 0$ such that

$$0 = AX + XA^{\mathrm{T}} + B_0B_0^{\mathrm{T}} + \sum_{i=1}^{n} \mathrm{tr}\,(Z_i)B_iR_iB_i^{\mathrm{T}},$$

$$Z_j = C_jXC_j^{\mathrm{T}}$$

Using the Kronecker product, this matrix equation can be expressed as

$$M\begin{bmatrix}\mathrm{vec}\,(X)\\ \mathrm{vec}\,(Z_1)\\ \vdots\\ \mathrm{vec}\,(Z_n)\end{bmatrix} = \begin{bmatrix}-\mathrm{vec}\,(B_0B_0^{\mathrm{T}})\\ 0\\ \vdots\\ 0\end{bmatrix}$$

where

$$M = \begin{bmatrix} I\otimes A + A\otimes I & \mathrm{vec}\,(B_1R_1B_1^{\mathrm{T}})\mathrm{vec}^{\mathrm{T}}(I) & \dots & \mathrm{vec}\,(B_nR_nB_n^{\mathrm{T}})\mathrm{vec}^{\mathrm{T}}(I)\\ -C_1\otimes C_1 & I & & \\ \vdots & & \ddots & \\ -C_n\otimes C_n & & & I\end{bmatrix}$$

Note that $\otimes$ in the above is the Kronecker product and vec is the stacked vector of a matrix. Since the system is asymptotically stable in the mean-square sense, $I\otimes A + A\otimes I$ is not a singular matrix. Therefore M is an invertible matrix. Hence, if there exist solutions for X and Z_is, they must be unique. This implies (iv) $\Rightarrow$ (i).

In the following, we will show (i) $\Rightarrow$ (iv). Considering A to be an asymptotically stable matrix, there must exist $X_i = X_i^{\mathrm{T}} \geq 0$ for $i \in \{0, 1, \dots, n\}$ such that

$$0 = AX_0 + X_0A^{\mathrm{T}} + B_0B_0^{\mathrm{T}}$$

$$0 = AX_i + X_iA^{\mathrm{T}} + \sigma_iB_iB_i^{\mathrm{T}}, \quad i \in \{1, 2, \dots, n\}$$

If (i) holds there exists a finite vector ζ whose ith element is $\zeta_i = \mathrm{tr}\,(Z_i) \in \mathbb{R}_+$. Construct a matrix of the following form

$$X_\sigma = X_0 + \sum_{i=1}^{n} \sigma_i\zeta_iX_i \geq 0$$

It is obvious that this X_σ is finite. Since $R_i \leq \sigma I$, for each $R \in \boldsymbol{R}^\sigma$ there must exist

$$X_R \leq X_\sigma$$

such that

$$0 = AX_R + X_RA^{\mathrm{T}} + B_0B_0^{\mathrm{T}} + \sum_{i=1}^{n} \mathrm{tr}\,(Z_{R_i})B_iR_iB_i^{\mathrm{T}}$$

with

$$Z_{R_i} = C_iX_RC_i^{\mathrm{T}}$$

Hence (i) implies (iv). □

Proof of Theorem 2: FSN performance implies that (1) is robust FSN stable or, $\forall R \in \boldsymbol{R}^\sigma$

$$\det\,(I - \mathtt{G}_R) > 0$$

and the worst-case performance $J < \gamma$ implies the output variance $\mathbf{E}_\infty[\|z_0\|^2]$ satisfies

$$1 - \mathbf{E}_\infty[\|z_0\|^2]\frac{1}{\gamma} > 0, \quad \forall w \in \mathcal{W}(\sigma, z)$$

which is equivalent to, for all $0 \leq r_0 \leq \frac{1}{\gamma}$, $R \in \boldsymbol{R}^\sigma$

$$\det\,[I - \mathtt{G}_R](1 - r_0\mathbf{E}_\infty[\|z_0\|^2]) > 0$$

Eliminating ζ in (4) and (2) or (3) and substituting the result into the above inequality leads to $R_\gamma \in \boldsymbol{R}_\gamma^\sigma$

$$\det\,[I - \hat{\mathtt{G}}_{R_\gamma}] > 0$$

where

$$\boldsymbol{R}_\gamma^\sigma = \left\{\text{block diag}\,(r_0I, R) : 0 \leq r_0 \leq \frac{1}{\gamma}, R \in \boldsymbol{R}^\sigma\right\}$$

which is equivalent to

$$\max_{R_\gamma \in \boldsymbol{R}_\gamma^\sigma} \bar{\lambda}(\hat{\mathtt{G}}_{\hat{R}_\gamma}) < 1.$$

Proof of Theorem 4: Let

$$H_i(K, \sigma, e) = \sum_{j=1}^{n} \alpha_j\sigma_j^{-1} + \beta\frac{\|T_{z_iw}(P, K)W^{1/2}(e, \sigma)\|_2^2}{e_i}$$

It is a well-known fact that if we change the order of min max to max min the following inequality holds

$$\min_\sigma \max_i H_i(K, \sigma, e) \geq \max_i \min_\sigma H_i(K, \sigma, e) \qquad (20)$$

Since $H_i(K, \sigma, e)$ can be written as a sum of a series of convex functions of σ_i

$$H_i(K, \sigma, e) = \sum_{j=1}^{n}\left\{\alpha_j\sigma_j^{-1} + \beta\sigma_j\frac{e_j}{e_i}\|T_{z_iw_j}(P, K)\|_2^2\right\}$$

Therefore the global minimum of $H_i(K, \sigma, e)$ with respect to σ can be achieved at those σs satisfying

$$\frac{\partial H_i(K, e, \sigma)}{\partial \sigma_j} = 0, \quad j \in \{1, \dots, n\}$$

or

$$\sigma = \sqrt{\frac{e_i}{\beta}}\left[\sqrt{\frac{\alpha_1}{e_1}}\frac{1}{\|T_{z_iw_1}(P, K)\|_2}\sqrt{\frac{\alpha_2}{e_2}}\frac{1}{\|T_{z_iw_2}(P, K)\|_2} \cdots \sqrt{\frac{\alpha_n}{e_n}}\frac{1}{\|T_{z_iw_n}(P, K)\|_2}\right]^{\mathrm{T}}$$

Inserting this optimal σ, we obtain

$$\max_i \min_\sigma H_i(K, \sigma, e) = \max_{1\leq i\leq n,\, e_i\neq 0} 2\sqrt{\beta}\sum_{j=1}^{n}\sqrt{\frac{\alpha_je_j}{e_i}}\|T_{z_iw_j}(P, K)\|_2$$

Assume σ^* and i^* are the optimal solutions for

$$\min_\sigma \max_i H_i(K, \sigma, e)$$

then σ^* must also be the optimal solution to solve

$$\min_{\sigma} H_{i^*}(K,\sigma,e)$$

i.e. there exists an $i^* \in \{1,\dots,n\}$ such that

$$\min_{\sigma}\max_{i} H_i(K,\sigma,e) = \sum_{j=1}^{n} 2\sqrt{\frac{\beta\alpha_j e_j}{e_{i^*}}}\|T_{z_i^* w_j}(P,K)\|_2$$

This implies

$$\min_{\sigma}\max_{i} H_i(K,\sigma,e) \le \max_{i}\min_{\sigma} H_i(K,\sigma,e) \qquad (21)$$

Combining (20) and (21) leads to

$$\min_{\sigma}\max_{i} H_i(K,\sigma,e) = \max_{i}\min_{\sigma} H_i(K,\sigma,e)$$

Hence we must have

$$L(K) = \min_{e,\sigma}\max_{i} H_i(K,\sigma,e)$$

Using this result leads to the theorem. □

Proof of Theorem 5: Let us first review the known results from the Perron–Frobenius theorem in Horn and Johnson (1985). Consider an $n\times n$ matrix A all of whose elements a_{ij} are positive. Then the Perron–Frobenius theorem says that

$$\min_{e\in\mathbb{R}^n_+}\ \max_{1\le i\le n,\, e_i\ne 0} \sum_{j=1}^{n} \frac{e_j}{e_i} a_{ij} = \bar\lambda(A)$$

Assume e is the eigenvalue associated with $\bar\lambda(A)$ satisfying $\sum_{i=1}^{n} e_i = 1$; then this e is the unique optimal solution to solve (A).

By Theorem 4 we have

$$L(K) = \min_{e\in\mathbb{R}^n_+}\ \max_{1\le i\le n,\, e_i\ne 0} 2\sqrt{\beta}\sum_{j=1}^{n}\sqrt{\frac{\alpha_j e_j}{e_i}}\|T_{z_i w_j}(P,K)\|_2$$

Hence, using the Perron–Frobenius theorem for this $L(K)$ leads to (i) and (ii). □

Proof of Theorem 6: Consider

$$S(K,\sigma) = \sum_{j=1}^{n}\alpha_j\sigma_j^{-1} + \beta\min_{e\in\mathbb{R}^n_+}\ \max_{1\le i\le n,\, e_i\ne 0}$$

$$\sum_{j=1}^{n}\frac{e_j}{e_i}\|T_{\hat z_j\hat w}(P,K)W_\gamma^{1/2}(e,\sigma)\|_2^2$$

It is not hard to see that $S(K,\sigma)$ is a continuous function of σ and a continuous functional of K. For a given σ, we have

$$N(\sigma) = S(K^*,\sigma), \quad \text{where } K^* = \arg\min_{K} S(K,\sigma)$$

and for a given K

$$L(K) = S(K,\sigma^*), \quad \text{where } \sigma^* = \arg\min_{\sigma} S(K,\sigma)$$

Hence the $\sigma - K$ iteration generates a sequence of controller K^ks and a sequence of noise-to-signal σ^ks such that

$$0 \le S(K^{k+1},\sigma^{k+1}) \le S(K^k,\sigma^{k+1}) \le S(K^k,\sigma^k)$$

Hence the sequence $S(K^k,\sigma^k)$ for $k = 1,2,\dots,\infty$ is a bounded-below monotonic decreasing sequence. There must exist a K^* and σ^* such that

$$\lim_{k\to\infty} K^k = K^*,\ \lim_{k\to\infty}\sigma^k = \sigma^*,\ \lim_{k\to\infty} S(K^k,\sigma^k) = S(K^*,\sigma^*)$$

□

Proof of Theorem 7: From Theorem 1 it is not hard to notice that the ϕ measure is monotonic with respect to σ_i for $i\in\{0,\dots,n\}$, i.e. if for $\hat\sigma$ and $\tilde\sigma$ satisfying

$$\hat\sigma_i \le \tilde\sigma_i, \quad i\in\{0,\dots,n\}$$

then

$$\phi(T_{\hat z\hat w}(P,K),\hat\sigma) \le \phi(T_{\hat z\hat w}(P,K),\tilde\sigma)$$

and this implies

$$\min_{K}\phi(T_{\hat z\hat w}(P,K),\hat\sigma) \le \min_{K}\phi(T_{\hat z\hat w}(P,K),\tilde\sigma)$$

Denote for $\sigma\in\mathbb{R}^n_+$

$$\psi = \min_{K}\phi\left(T_{\hat z\hat w}(P,K), \begin{bmatrix}\gamma^{-1}\\ \sigma\end{bmatrix}\right)$$

then

$$\min_{K}\phi\left(T_{\hat z\hat w}(P,K), \begin{bmatrix}\gamma^{-1}\psi^{-1}\\ \sigma\psi^{-1}\end{bmatrix}\right) = 1$$

If $\psi > 1$, then $\gamma^{-1} > \gamma^{-1}\psi^{-1}$ implies

$$\min_{K}\phi\left(T_{\hat z\hat w}(P,K), \begin{bmatrix}\gamma^{-1}\\ \sigma/\psi\end{bmatrix}\right) \ge 1$$

Hence if $\phi_0 > 1$, then the sequence ϕ_k for $k\in\{0,1,\dots,\infty\}$ is a monotonic sequence satisfying

$$1 \le \phi_{k+1} \le \phi_k$$

i.e. there exists a finite $\phi^* \ge 1$ such that $\lim_{k\to\infty}\phi_k = \phi^*$ and there exists a $\sigma^*\in\mathbb{R}^n_+$ satisfying

$$\lim_{k\to\infty}\sigma^k = \lim_{k\to\infty}\frac{\sigma^0}{\phi_0\phi_1\dots\phi_{k-1}} = \sigma^*$$

Now we want to prove the limit $\phi^* = 1$. Since for $j\in\{1,\dots,n\}$

$$\sum_{t=0}^{k-1}\ln(\phi_t) = \ln(\sigma_j^0) - \ln(\sigma_j^k)$$

and at least there exists a j such that $\sigma_j^* > 0$. For this j

$$\lim_{k\to\infty}\ln(\sigma_j^k) = \ln(\sigma_j^*) \text{ is finite}$$

hence the series

$$\sum_{t=0}^{k-1} \ln(\phi_t)$$

converges. A necessary condition for this series to converge is

$$\lim_{t\to\infty} \ln(\phi_t) = 0$$

which implies

$$\phi^* = \lim_{t\to\infty} \phi_t = 1$$

Therefore the theorem is true. □

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