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Integrating closed-loop identification and control using control energy sensitivity

JIANBO LU†* and ROBERT E. SKELTON‡

This paper extracts a plant model from an identified closed-loop system model, and proposes an iterative procedure to find high performance controllers. The algorithm uses *control energy sensitivity* to connect closed-loop modelling and control design. This sensitivity describes the gradient of the consumed control energy with respect to the achieved output covariance. Hence it captures the relative importance of each output channel in the closed-loop system behaviour. Therefore the identification incorporating *control energy sensitivity* can characterize models to generate a controller for achieving better closed-loop performance. The procedure is demonstrated in a structural control problem.

1. Introduction

Synthesis of a model-based feedback control requires a model of the plant. Using modern control strategies, the order of the controller is usually determined by the order of the model. Models need to be appreciated for the particular inputs to be applied. Hence actuator features also influence what types of modelling should be used. For example, large actuation range implies the need of a non-linear model to reflect the system physics, while a linear model is usually valid for small actuation range. This is just one indication that plant modelling and control design need to be considered together, see Skelton (1989) for more specific. The interdependence between modelling and control plays important roles in designing low order controllers to achieve high system performances, or in meeting very stringent performance requirements.

In a separated modelling and control approach, the model used for control development is deduced from the open-loop responses and is not affected by the controller at all. This is called *open-loop modelling*. The open-loop modelling may not be suitable for high performance control system design. This has created quite an interest in closed-loop identification, see Liu and Skelton (1993), Hjalmarsson *et al.* (1995) and Zang *et al.* (1995). The *closed-loop modelling* studied here deduces a plant model from the closed-loop system response, reflecting controlled system information.

To produce a model which is good at the control-relevant frequencies, weighted frequency domain identification is used in Zang *et al.* (1995). In the time domain, Skelton and Shi (1996) provides a method using a control-relevant weight in open-loop identification. The method in Skelton and Shi (1996) is not true closed-loop modelling, however it is possible for it to be extended to closed-loop modelling. This extension can be found in Lu and Skelton (1998), where it has been successfully demonstrated in a civil structure control application. The contribution of the current paper is to improve the method in Lu and Skelton (1998) using a control-relevant weight to closed-loop modelling. The weight used here is obtained from the control design step. While the physical meaning of a similar weight is not clear in Lu and Skelton (1998), this paper proves that it is the sensitivity of the control energy with respect to the achieved system covariance. We will call this weight *control energy sensitivity* or CES.

This paper is organized as follows. Section 2 describes the covariance control design method, in particular the means to choose feasible covariance bound, the procedure to find minimum achieved covariance and the physical meaning of the Kuhn–Tucker weight obtained in covariance control design (the gradient of the control energy with respect to the achieved covariance). Section 3 studies cloosed loop identification by accommodating the *control energy sensitivity* generated from the last control design step, and extracting a plant model from the closed loop model. Iteration between closed-loop modelling and control is discussed in §4. A numerical example is included in §5. Section 6 offers the conclusion.

The following notations are used in this paper. $\mathbf{E}(\cdot)$ and $\mathbf{E}_{\infty}(\cdot)$ denote the expectation operator and the steady state expectation operator of discrete time but continuously-valued stochastic processes. $(\cdot)^{\mathrm{T}}$ and $(\cdot)^{+}$ denote the transpose and the Penrose–Moore generalized inverse of a matrix $(\cdot)$. $\otimes$ denotes the Kronecker product. $\mathrm{tr}(\cdot)$ means the sum of all the diagonal elements of a matrix $(\cdot)$. SVD is short for the singular value decomposition. For a matrix M, $M > 0$ (≥ 0) implies M is a positive definite (semi-positive definite) matrix. A sequence of positive definite matrices $\bar{Z} = \{\bar{Z}^1, \bar{Z}^2, \ldots\}$ is said to approach infinity, or $\bar{Z} \to \infty$, if for any given integer $k > 0$, there exists a

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*Author for correspondence. e-mail: jlu10@ford.com

† Research and Vehicle Technology, Ford Motor Company, Building #5, Mail Drop 5020, Dearborn, MI 48124, USA.

‡ Department of Applied Mechanics and Engineering Sciences, University of California at San Diego, La Jolla, CA 92093-0411, USA.

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positive definite matrix $Z^k > kI$, such that $\bar{Z}^k > Z^k$ holds, where I is a unit matrix.

2. Covariance control with control energy sensitivity

Consider the discrete time plant G

$$\left.\begin{aligned} x_{k+1} &= Ax_k + B_1 w_k + B_2 u_k \\ z_k &= Cx_k \\ y_k &= C_y x_k + s_k \end{aligned}\right\} \qquad (1)$$

where $x_k \in \mathbb{R}^{n_x}$, $z_k \in \mathbb{R}^{n_z}$, $y_k \in \mathbb{R}^{n_y}$, $u_k \in \mathbb{R}^{n_u}$, $w_k \in \mathbb{R}^{n_w}$ and $s_k \in \mathbb{R}^{n_y}$ are the system state, the controlled variable, the sensor measurement, the actuator variable, the plant disturbance and the sensor noise. We assume that (1) is stabilizable and detectable. A strictly proper controller K for plant (1) has the form

$$\left.\begin{aligned} x_{c_{k+1}} &= A_c x_{c_k} + B_c y_k \\ u_k &= C_c x_{c_k} \end{aligned}\right\} \qquad (2)$$

The state space description of the closed-loop generated from this controller can be expressed as

$$\left.\begin{aligned} x_{clk+1} &= A_{cl} x_{clk} + B_{cl} w_{clk} \\ z_k &= [C \quad 0] x_{clk} \\ u_k &= [0 \quad C_c] x_{clk} \end{aligned}\right\} \qquad (3)$$

where x_{cl} is the closed loop system states and w_{cl} is the disturbance vector. They are defined as

$$x_{cl} = \begin{bmatrix} x \\ x_c \end{bmatrix}, \quad w_{cl} = \begin{bmatrix} w \\ s \end{bmatrix}$$

and the system matrices A_{cl} and B_{cl} can be expressed as

$$A_{cl} = \begin{bmatrix} A & B_2 C_c \\ B_c C_y & A_c \end{bmatrix}, \quad B_{cl} = \begin{bmatrix} B_1 & 0 \\ 0 & B_c \end{bmatrix}$$

Let C be partitioned as

$$C = [C_1^{\mathrm{T}} \quad C_2^{\mathrm{T}} \quad \cdots \quad C_n^{\mathrm{T}}]^{\mathrm{T}}$$

with the ith element of row dimension m_i and $\sum_{i=1}^{n} m_i = n_z$. Define

$$z_i = C_i x$$

and

$$\boldsymbol{P} \triangleq \{\text{block diag}(P_1, P_2, \ldots, P_n): P_i \in \mathbb{R}^{m_i \times m_i}$$
$$P_i = P_i^{\mathrm{T}} \geq 0, i = 1, 2, \ldots, n\}$$

Skelton (1988) studied the so-called output covariance control (OCC), which is a procedure to iteratively update a weight Q such that a controller delivers a closed-loop system whose output covariance is constrained by a given covariance bound. This problem can be summarized as the following.

Output covariance control problem: For a given

$$\bar{Z} = \text{block diag}(\bar{Z}_1, \bar{Z}_2, \ldots, \bar{Z}_n) \in \boldsymbol{P}$$

find a controller K of form (2) for the plant G in (1) to solve the constrained optimization

$$J_{sopt}(\bar{Z}) = \min_K \{\mathbf{E}_\infty[u^{\mathrm{T}} u]: \text{subject to}$$
$$\mathbf{E}_\infty[z_i z_i^{\mathrm{T}}] \leq \bar{Z}_i, i = 1, 2, \ldots, n\} \qquad (4)$$

Since not all $\bar{Z} \in \boldsymbol{P}$ can deliver an optimal controller for the OCC problem, it is natural to ask when the given covariance bound $\bar{Z} \in \boldsymbol{P}$ is feasible for the optimization.

Definition 1: A $\bar{Z} \in \boldsymbol{P}$ is said to be OCC-feasible if $J_{sopt}(\bar{Z})$ is finite and the optimal controller K_{sopt}, defined as

$$J_{sopt}(\bar{Z}) = \arg\min_K \{\mathbf{E}_\infty[u^{\mathrm{T}} u]: \text{subject to}$$
$$\mathbf{E}_\infty[z_i z_i^{\mathrm{T}}] \leq \bar{Z}_i, i = 1, 2, \ldots, n\} \qquad (5)$$

delivers a closed loop system with output covariance bounded by $\bar{Z}$.

From Skelton (1988) and Zhu *et al.* (1997), $\bar{Z} \in \boldsymbol{P}$ is OCC-feasible if and only if there exists $\tilde{X}, X, Y \geq 0$, $Q \in \boldsymbol{P}$ satisfying the following matrix equalities and inequalities:

(i) Riccati equations

$$Y = A^{\mathrm{T}} YA - A^{\mathrm{T}} YB_2 (I + B_2^{\mathrm{T}} YB_2)^{-1} B_2^{\mathrm{T}} YA + \sum_{i=1}^{n} C_i^{\mathrm{T}} Q_i C_i \qquad (6)$$

$$\tilde{X} = A\tilde{X}A^{\mathrm{T}} - A\tilde{X}C_y^{\mathrm{T}} (S + C_y \tilde{X} C^{\mathrm{T}} y)^{-1} C_y \tilde{X} A^{\mathrm{T}} + B_1 W B_1^{\mathrm{T}} \qquad (7)$$

where W and S are the covariances of the disturbance w_k and the sensor noise is s_k;

(ii) Lyapunov equation

$$\begin{aligned} X = {} & [I - B_2 (I + B_2^{\mathrm{T}} YB_2)^{-1} B_2^{\mathrm{T}} Y] AXA^{\mathrm{T}} \\ & \times [I - B_2 (I + B_2^{\mathrm{T}} YB_2)^{-1} B_2^{\mathrm{T}} Y]^{\mathrm{T}} \\ & + A\tilde{X} C_y^{\mathrm{T}} (S + C_y \tilde{X} C_y^{\mathrm{T}})^{-1} C_y \tilde{X} A^{\mathrm{T}} \end{aligned} \qquad (8)$$

(iii) Kuhn–Tucker condition

$$Q[C(X + \tilde{X})C^{\mathrm{T}} - \bar{Z}] = 0 \qquad (9)$$

where Q is called the Kuhn–Tucker parameter or Kuhn–Tucker weight;

(iv) Performance bound

$$C(X + \tilde{X})C^{\mathrm{T}} \leq \bar{Z} \qquad (10)$$

Assume $\bar{Z}$ is OCC-feasible. Let $\tilde{X}, X, Y \geq 0$, $Q \in \boldsymbol{P}$ be the matrices satisfying (6)–(10). Then the optimal controller (5) will have the system matrix triple

$$\left.\begin{aligned} B_c(\bar{Z}) &= A\tilde{X}C_y^{\mathrm{T}}(V + C_y\tilde{X}C_y^{\mathrm{T}})^{-1} \\ C_c(\bar{Z}) &= -(I + B_2^{\mathrm{T}}YB_2)^{-1}B_2^{\mathrm{T}}YA \\ A_c(\bar{Z}) &= A + B_2C_c - B_cC_y \end{aligned}\right\} \quad (11)$$

Notice that A_c, B_c and C_c are functions of the covariance bound $\bar{Z}$. We should call the controller K_{sopt} defined in (11) the *$\bar{Z}$-suboptimal controller*, since $\bar{Z}$ is not intended to be achieved by the output covariance of the closed loop system.

In order to find the unique matrices $\tilde{X}, X, Y \geq 0$, $Q \in \boldsymbol{P}$ satisfying (6)–(10) (and hence the controller K_{sopt}), an iterative but convergent procedure has been studied in Skelton (1988) and Zhu *et al.* (1997). The following is a summary of this procedure.

OCC Algorithm 1: Given an OCC-feasible output covariance bound

$$\bar{Z} = \text{block diag}\,(\bar{Z}_1, \bar{Z}_2, \ldots, \bar{Z}_n) \in \boldsymbol{P}$$

a tolerance $\epsilon > 0$, an initial weight

$$Q^0 = \text{block diag}\,(Q_1^0, Q_2^0, \ldots, Q_n^0) \in \boldsymbol{P}$$

and positive scalars α and β such that

$$\alpha > 0, \quad 0 < \beta < 1$$

Step 1. Compute $\tilde{X}$ from (7).

Step 2. Compute $Y \geq 0$ by solving

$$Y = A^{\mathrm{T}}YA - A^{\mathrm{T}}YB_2(I + B_2^{\mathrm{T}}YB_2)^{-1}B_2^{\mathrm{T}}YA + \sum_{i=1}^{n} C_i^{\mathrm{T}}Q_i^0C_i$$

For this Y, compute X from (8).

Step 3. Compute

$$Z_i = C_i(X + \tilde{X})C_i^{\mathrm{T}} \quad i = 1, 2, \ldots, n$$

If the following holds

$$\max_{1\leq i\leq n} \|Q_i^0(Z_i - \bar{Z}_i\| \leq \epsilon$$

then stop (where $\|\cdot\|$ could be any matrix norm), and compute the controller system matrices $A_c(\bar{Z})$, $B_c(\bar{Z})$ and $C_c(\bar{Z})$ using formula (11), which assembly the $\bar{Z}$-suboptimal controller K_{sopt}. Otherwise, compute

$$Q = \beta Q^0 + (1 - \beta)\mathcal{P}[Q^0 + \alpha(Z - \bar{Z})]$$

and set $Q^0 = Q$, and go to Step 2. Notice that

$$Q = \text{block diag}\,(Q_1, Q_2, \ldots, Q_n) \in \boldsymbol{P}$$

Remark: Notice that $\mathcal{P}[\cdot]$ in Step 3 is a projection operator

$$P[M] = \begin{cases} 0 & \text{if } M \leq 0 \\ V_+E_+V_+^{\mathrm{T}} & \text{otherwise} \end{cases}$$

where

$$M = V_-E_-V_-^{\mathrm{T}} + V_+E_+V_+^{\mathrm{T}}$$

is the Schur decompositon of a symmetric matrix M, and E_+ includes all the strictly positive eigenvalues of M.

Notice that although the OCC feasibility conditions in (6)–(10) can be checked through the OCC algorithm, it is mainly used for finding the optimal controller K_{sopt}. If we apply the OCC algorithm and do not get convergent solutions, we can claim that $\bar{Z}$ is not OCC-feasible. However this approach is not preferable. It is better to know whether $\bar{Z}$ is OCC-feasible before doing OCC iteration. The following theorem shows that an OCC-feasible $\bar{Z}$ can be constructed solely from the system plant (the open loop system).

Let Z_i for $i = 1, 2, \ldots, n$ be the achieved output covariance of the ith output z_i in the closed loop system, generated from the $\bar{Z}$-suboptimal control K_{sopt}, then Z_i is a function of the covariance bound $\bar{Z}$ and we denote such functions through the definition

$$Z = \text{block diag}\,(Z_1, Z_2, \ldots, Z_n) \triangleq \mathcal{C}(\bar{Z}) \quad (12)$$

Theorem 1: *Let $Z \in \boldsymbol{P}$ be the achieved covariance defined in* (12) *for the closed loop system generated from the $\bar{Z}$-suboptimal controller K_{sopt} for a given $\bar{Z} \in \boldsymbol{P}$. Then the following results hold:*

(i) *The achieved performance Z is bounded by*

$$Z \leq \lim_{\bar{Z}\to\infty} \mathcal{C}(\bar{Z})$$

(ii) *If A is discrete-time stable, i.e. $\rho(A) < 1$, then*

$$\lim_{\bar{Z}\to\infty} \mathcal{C}(\bar{Z}) = CX_{ol}C^{\mathrm{T}}$$

where X_{ol} is the state covariance of the plant satisfying

$$X_{ol} = AX_{ol}A^{\mathrm{T}} + B_1WB_1^{\mathrm{T}}$$

otherwise if A is not discrete time stable, then

$$\lim_{\bar{Z}\to\infty} \mathcal{C}(\bar{Z}) = \infty$$

(iii) *The following holds*

$$Z \geq \underline{Z}$$

where

$$\underline{Z} = \text{block diag}\,(\underline{Z}_1, \underline{Z}_2, \ldots, \underline{Z}_n),$$

and

$$\underline{Z}_i = C_i \tilde{X} C_i^{\mathrm{T}}, \quad i = 1, 2, \ldots, n$$

with $\tilde{X}$ satisfying (7).

(iv) *If A is discrete time asymptotically stable, an OCC-feasible $\bar{Z}$ can be constructed as*

$$\bar{Z} = \kappa \underline{Z} + (1-\kappa) Z_{ol} \tag{13}$$

for some κ satisfying $0 \le \kappa < 1$.

If A is not discrete time asymptotically stable, an OCC-feasible $\bar{Z}$ can be constructed as

$$\bar{Z} = \underline{Z}/\kappa \tag{14}$$

for some κ satisfying $0 < \kappa < 1$.

Proof: (i) is obvious. Let us consider (ii). If $\bar{Z}$ is large enough (for example, $\bar{Z} \to \infty$), $Z < \bar{Z}$ is always true. Hence the Kuhn–Tucker weight Q in (9) must be zero. Therefore (6) satisfies

$$Y = A^{\mathrm{T}} Y A - A^{\mathrm{T}} Y B_2 (I + B_2^{\mathrm{T}} Y B_2)^{-1} B_2^{\mathrm{T}} Y A \tag{15}$$

Assume there is a $Y \ge 0$ satisfying (15), then we must have

$$Y = -\sum_{k=1}^{\infty} A^{\mathrm{T}k+1} Y B_2 (I + B_2^{\mathrm{T}} Y B_2)^{-1} B_2^{\mathrm{T}} Y A^{k+1} \le 0$$

This implies that $Y = 0$ is the only solution of (15). Hence (8) is simplified to

$$X = AXA^{\mathrm{T}} + A\tilde{X}C_y^{\mathrm{T}}(S + C_y \tilde{X} C_y^{\mathrm{T}})^{-1} C_y \tilde{X} A^{\mathrm{T}}$$

Adding this equation to (7) leads to

$$X + \tilde{X} = A(X + \tilde{X})A^{\mathrm{T}} + B_1 W B_1^{\mathrm{T}}$$

Therefore the first part of (ii) is true. The second part of (ii) is obvious, due to the fact that the state covariance X_{ol} of the plant is unbounded, or say $X_{ol} \to \infty$.

(iii) is obvious due to the fact that $X \ge 0$, i.e.

$$Z_i = C_i(X + \tilde{X})C_i^{\mathrm{T}} \ge C_i \tilde{X} C_i^{\mathrm{T}}$$

Let us prove (iv). (i)–(iii) imply that an OCC-feasible $\bar{Z}$ falls within the bounds

$$\underline{Z} \le \bar{Z} \le Z_{ol} \tag{16}$$

If the plant is stable, the following positive definite matrix satisfies (16) for κ satisfying $0 \le \kappa < 1$

$$\bar{Z}(\kappa) = \kappa \underline{Z} + (1-\kappa) Z_{ol}$$

Notice that $\mathcal{C}(\bar{Z}) \le \bar{Z}(\kappa)$ is equivalent to

$$CXC^{\mathrm{T}} \le (1-\kappa) C(X_{ol} - \tilde{X}) C^{\mathrm{T}}$$

and a sufficient condition for the above is

$$X \le (1-\kappa)(X_{ol} - \tilde{X})$$

Due to a property of Riccati equations, there is a $Q \in \boldsymbol{P}$ such that the Y computed from (6) yields an $X = X^{\mathrm{T}} > 0$ such that the above inequality for X is satisfied for some κ, see Skelton *et al.* (1997) for detail. Notice that, there is no constraint on Q as soon as it belongs to $\bar{P}$. If A is not discrete time asymptically stable, the following matrix satisfies (16) for each $0 < \kappa < 1$

$$\bar{Z} = \underline{Z}/\kappa \tag{17}$$

Similarly, there is an X satisfying the sufficient condition for (17)

$$X \le \frac{1-\kappa}{\kappa} \tilde{X}$$

and (8) with some choice of $Q \in \bar{P}$, that delivers some $Y = Y^{\mathrm{T}} > 0$ satisfying (6). Therefore the claim is true. □

Notice that the optimal performance of interest is different from (4). Instead of minimizing the control energy with respect to the output covariance constraint, we are interested in minimizing the output variance with respect to a control covariance bound as in

$$\min_K \{\mathbf{E}_\infty[z_i^{\mathrm{T}} R z_i]\text{: subject to } \mathbf{E}_\infty[uu^{\mathrm{T}}] \le \bar{U}\} \tag{18}$$

Since (18) does not have the nice feature of choosing a feasible bound $\bar{U}$ as in Theorem 1 and other features embedded in optimization (4), much of the attention has been focused on (4). However, we might want to trade a tight output performance for an increased control energy in order to meet stringent performance requirements. Now the question is whether we can find a controller to achieve as small a covariance as possible through the OCC problem (4). The following discussion shows that an iterative OCC procedure can achieve this.

Generally speaking, there is a gap between the output covariance $\mathcal{C}(\bar{Z})$ and the bound $\bar{Z}$, that is

$$\bar{Z} - \mathcal{C}(\bar{Z}) \ge 0$$

If $\bar{Z}$ is achievable by an OCC controller, the above gap is diminished. Let us consider the optimization

$$J_{\mathrm{opt}}(\bar{Z}) = \min_K \{\mathbf{E}_\infty[u^{\mathrm{T}} u]\text{: subject to } \mathcal{C}(\bar{Z}) = \bar{Z}\} \tag{19}$$

and its optimal controller K_{opt}

$$K_{\mathrm{opt}} = \arg\min_K \{\mathbf{E}_\infty[u^{\mathrm{T}} u]\text{: subject to } \mathcal{C}(\bar{Z}) = \bar{Z}\} \tag{20}$$

We should call such K_{opt} the $\bar{Z}$-optimal controller. The following theorem shows how to use a series of $\bar{Z}$-suboptimal controllers to approach the $\bar{Z}$-optimal controller.

Theorem 2: *Let $\bar{Z}^0 \in \boldsymbol{P}$ be OCC-feasible. Consider the following iteration from the OCC algorithm*

$$\bar{Z}^{k+1} = \mathcal{C}(\bar{Z}^k), \quad k = 0, 1, \ldots \tag{21}$$

i.e. the $(k+1)$th controller K^k_{sopt} is $\mathcal{C}(\bar{Z}^k)$-suboptimal. Denoted as Q^k the kth *Kuhn–Tucker weight and $J^k_{sopt}(\bar{Z}^k)$ the* kth *optimal control energy corresponding to the* kth *controller K^k_{sopt} and the* kth *covariance bound $\bar{Z}^k$. Then we have the following:*

(i) *All $\bar{Z}^k$ are OCC-feasible.*

(ii) *The iteration* (21) *converges. That is, there exist a weight $Q^* \in \boldsymbol{P}$, a controller K^* and a $\bar{Z}^* \in \boldsymbol{P}$ such that*

$$\lim_{k\to\infty} \bar{Z}^k = \bar{Z}^*$$

$$\lim_{k\to\infty} Q^k = Q^*$$

$$\lim_{k\to\infty} K^k_{sopt} = K^*$$

and

$$\bar{Z}^* = \mathcal{C}(\bar{Z}^*)$$

where $\mathcal{C}(\bar{Z}^*)$ is the output covariance of the closed loop system generated from the controller K^*. K^* is the $\bar{Z}$-optimal controller, or say

$$K_{opt} = K^*$$

(iii) *For any k, we have*

$$J_{opt}(\bar{Z}^*) = \lim_{k\to\infty} J_{sopt}(\bar{Z}^k) = \sup_k J_{sopt}(\bar{Z}^k) \geq J_{sopt}(\bar{Z}^k)$$

(iv) *The Q^* obtained in* (ii) *is strictly positive definite.*

Proof: Assume (i) is not true, then there exists a k such that $\bar{Z}^k$ is not OCC-feasible and $\bar{Z}^{k-1}$ is OCC-feasible. However

$$\bar{Z}^k = \mathcal{C}(\bar{Z}^{k-1})$$

implies that there is an OCC controller to achieve $\bar{Z}^k$ as a covariance bound. This leads to a contradiction, hence (i) must be true.

Let us consider (ii). From the definition of $\mathcal{C}(\bar{Z})$, the following is true

$$\mathcal{C}(\bar{Z}) \leq \bar{Z}$$

Hence the iteration (21) generates a matrix sequence satisfying

$$\bar{Z}^0 \geq \bar{Z}^1 \geq \cdots \geq 0$$

This is a bounded below, monotonically decreasing sequence (with respect to the matrix ordering operation defined by the semi-positive definite operation $\geq$). There must exist a matrix $\bar{Z} \in \boldsymbol{P}$ such that

$$\lim_{k\to\infty} \bar{Z}^k = \bar{Z} = \inf_k \bar{Z}^k = \mathcal{C}\Big(\lim_{k\to\infty} \bar{Z}^k\Big) = \mathcal{C}(\bar{Z})$$

Since for each $\bar{Z}^k$, K^k_{sopt} and Q^k generated from OCC algorithm are unique. Hence both sequences of K^k_{sopt} and Q^k have to converge. Notice that the convergence implies

$$K^* = \arg\min_K \{\mathbf{E}_\infty[u^\mathrm{T}u]\text{: subject to } \mathcal{C}(\bar{Z}^*) = \bar{Z}^*\}$$

Then (ii) is true.

Let us consider (iii). (i) implies that all $\bar{Z}^k$ and $\bar{Z}^*$ are OCC-feasible, hence $J_{opt}(\bar{Z}^*)$ is finite. Also notice that

$$\bar{Z}^{k+1} \leq \bar{Z}^k, \quad k = 0, 1, \ldots$$

therefore

$$\min_K \{\mathbf{E}_\infty[u^\mathrm{T}u]\text{: subject to } \mathcal{C}(\bar{Z}^{k+1}) \leq \bar{Z}^{k+1}\}$$
$$\geq \min_K \{\mathbf{E}_\infty[u^\mathrm{T}u]\text{: subject to } \mathcal{C}(\bar{Z}^k) \leq \bar{Z}^k\}$$

Hence $J_{sopt}(\bar{Z}^k)$ for $k = 0, 1, \ldots$ is a monotonically increasing sequence, and $J_{sopt}(\bar{Z}^*)$ is its achievable upper bound. That is

$$J_{sopt}(\bar{Z}^*) = \sup_k J_{sopt}(\bar{Z}^k)$$

Therefore the claim is true.

Let us consider (iv). Assume Q^* is not strictly positive definite, i.e. $\bar{Z}^*$ is not an active constraint for the OCC probem. Then we must have

$$\mathcal{C}(\bar{Z}^*) \leq \bar{Z}^* \tag{22}$$

where $\mathcal{C}(\bar{Z}^*)$ is generated from the controller K^*. Equation (22) further implies that there exists a large enough integer N, such that

$$\bar{Z}^{N+1} = \mathcal{C}(Z^N) \leq \bar{Z}^*$$

This contradicts with the fact that $\bar{Z}^*$ is the lower-bound of the sequence $\bar{Z}^k$ for $k = 0, 1, \ldots, \infty$. Hence (iii) must hold. Therefore, the claims in the theorem are all true. □

The Kuhn–Tucker weight Q obtained before is related to the control energy and the performance bound $\bar{Z}$. Let $U(\bar{Z})$ be the optimal control energy computed for the closed-loop system generated from the $\bar{Z}$-suboptimal controller K_{sopt}, i.e.

$$U(\bar{Z}) = \mathbf{E}_\infty[u^\mathrm{T}u]$$

then the following theorem holds.

Theorem 3: *The optimal control energy $U(\bar{Z})$ of the $\bar{Z}$-suboptimal controller K_{sopt} is a monotonically decreasing function of the covariance bound $\bar{Z}$ and the decreasing rate is*

$$\frac{\partial U(\bar{Z})}{\partial \bar{Z}_i} = -Q_i$$

for $i = 1, 2, \ldots, n$. Where Q_i is the ith *block matrix in the Kuhn–Tucker weight Q generated from the OCC algorithm, which obeys* (9). *If $\bar{Z}$ is not achievable (i.e. it is an* *upper bound), then through the iteration considered in Theorem 2, we have*

$$\lim_{k\to\infty} \frac{\partial U(Z^k)}{\partial \bar{Z}_i^k} = -\lim_{k\to\infty} Q_i^k$$

Proof: By Theorem 2, given any OCC-feasible $\bar{Z} \in \boldsymbol{P}$, there exists a $\bar{Z}^* \in \boldsymbol{P}$ such that the OCC problem has an optimal solution K^*, which is a regular solution together with an associated Kuhn–Tucker multiplier $Q^* \in \boldsymbol{P}$, and satisfies the sufficient condition for a minimum with respect to the following constrained optimization problem

$$\min_K \{\mathbf{E}_\infty[u^{\mathrm{T}}u]\text{: subject to } \mathbf{E}_\infty[z_i z_i^{\mathrm{T}}] = \bar{Z}_i^*, \quad i = 1, 2, \ldots, n\} \quad (23)$$

By the sensitivity theorem in § 10.7 of Luenberger (1984) for every $\bar{Z} \in \boldsymbol{P}$ in a region containing $\bar{Z}^*$ there is a controller $K(\bar{Z})$ depending continuously on $\bar{Z}$ such that it is a minimum of (23) and $K(\bar{Z}^*) = K^*$. Notice that using Kuhn–Tucker multiplier, the cost function in (23) can be expressed as

$$\mathbf{E}_\infty[u^{\mathrm{T}}u] + \sum_{i=1}^{n} \mathrm{tr}\,[Q_i(Z_i - \bar{Z}_i)]$$

and the partial derivative of this with respect to $\bar{Z}_i$ is $-Q_i^{\mathrm{T}}$. Hence from the aforementioned sensitivity theorem, the following is true

$$\left.\frac{\partial U(\bar{Z})}{\partial \bar{Z}_i}\right|_{\bar{Z}=\bar{Z}^*} = -Q_i^{*\mathrm{T}}$$

Hence the claim follows. The proof of the rest of the theorem can be obtained by considering the uniqueness of the limit

$$\lim_{k\to\infty} \bar{Z}_i^k, \quad \lim_{k\to\infty} Q_i^k, \quad \lim_{k\to\infty} U(\bar{Z}^k)$$

and Theorem 2. □

The ith bock element Q_i of Q is the ith multiplier used in the Kuhn–Tucker condition of the constrained optimization problem (see Luenberger (1984) for detail). If the ith constraint Z_i is active, the associated Q_i has to be strictly positive definite; if it is inactive, the associated Q_i is semi-positive definite. In the former, the ith covariance constraint is achieved, hence the corresponding channel is harder to be controlled than those channels whose covariance constraints are inactive.

Furthermore, Theorem 3 implies that for the same amount of performance improvement, a large Q will cause a large increase of control energy, i.e. Q measures how sensitive the control energy with respect to performance requirement variation. For this reason, Q should be called the *control energy sensitivity*. The block elements Q_i's of Q weight the relative importance of variables z_i's in the closed-loop performance. Large Q_i means the ith output channel needs to consume a large amount of control energy so as to achieve the performance requirement. Quantitatively, we can order the importance of the system output channels in the closed loop behaviour by the index

$$I_i = \frac{\|Q_i\|}{\|Q\|}, \quad i = 1, 2, \ldots, n$$

That is, larger I_i's correspond to performance variables which are more important in achieving a control goal (more control energy needs to be consumed here for a small performance improvement).

3. Closed-loop modelling

A typical control system consists of a plant, a control computer in which a control algorithm is running, actuating hardware, sensing hardware, analog-to-digital (A/D) converter and digital-to-analog (D/A) converter. There are noises entering each of these components. For our interest, the actual plant, the involved hardware, the D/A converter and the A/D converter are all assembled together to form an augmented plant G. The control algorithm K is pulled out as shown in figure 1, where $w \in \mathbb{R}^{n_u}$ and $s \in \mathbb{R}^{n_y}$ characterize the noise inputs into the system; w, s, v_u and v_y are independent discrete-time white noise processes; and v_u and v_y are used for system identification or system modelling purposes.

Let $T(G, K)$ denote the closed-loop system in figure 1 relating $\nu + \omega$ to ζ with

$$\nu = \begin{bmatrix} v_u \\ v_y \end{bmatrix}, \quad \omega = \begin{bmatrix} w \\ s \end{bmatrix}, \quad \zeta = \begin{bmatrix} u \\ y \end{bmatrix}$$

A state space model for the closed-loop system using the response ζ with respect to the white noise input $\omega + \nu$ is expected. The identification method used here finds a linear state space model to match the date set

$$\mathbf{D}_q \triangleq \{R_{\zeta\zeta_i}, R_{\zeta\nu_i}, i = 0, 1, \ldots, q-1\}$$

where q is a chosen integer, and $R_{\zeta\zeta_i}$ and $R_{\zeta\nu_i}$ are computed from the noise response for $i = 0, 1, \ldots, q-1$, that is

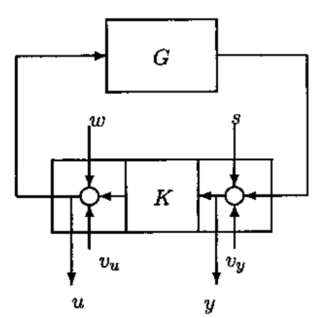

Figure 1. Setup for closed-loop identification.

$$R_{\zeta\zeta_i} \triangleq \lim_{k\to\infty} \mathbf{E}[\zeta_{k+1}\zeta_k^{\mathrm{T}}]$$

$$R_{\zeta\nu_i} \triangleq \lim_{k\to\infty} \mathbf{E}[\zeta_{k+1}\nu_k^{\mathrm{T}}]$$

A linear model thus obtained is called a q-Markov COVER (qMC) as in King *et al.* (1988) and Skelton (1988). In this section, we will use this method to obtain the closed loop system description.

Let us construct Toeplitz matrices

$$\left.\begin{aligned} \mathsf{R}_{\zeta\zeta} &\triangleq \begin{bmatrix} R_{\zeta\zeta_0} & R^{\mathrm{T}}_{\zeta\zeta_1} & \cdots & R^{\mathrm{T}}_{\zeta\zeta_{q-1}} \\ R_1 & R_0 & \cdots & R^{\mathrm{T}}_{\zeta\zeta_{q-2}} \\ \vdots & \vdots & \ddots & \vdots \\ R_{\zeta\zeta_{q-1}} & R_{\zeta\zeta_{q-2}} & \cdots & R_{\zeta\zeta_0} \end{bmatrix} \\ \mathsf{R}_{\zeta\nu} &\triangleq \begin{bmatrix} R_{\zeta\nu_0} & 0 & \cdots & 0 \\ R_{\zeta\nu_1} & R_{\zeta\nu_0} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ R_{\zeta\nu_{q-1}} & R_{\zeta\nu_{q-2}} & \cdots & R_{\zeta\nu_0} \end{bmatrix} \end{aligned}\right\} \qquad (24)$$

and denote the covariances of v_u and v_y as V_u and V_y, and the covariances of w and s as W and S. Define the covariances of ω and ν as

$$\Omega = \begin{bmatrix} W & 0 \\ 0 & S \end{bmatrix}, \quad V = \begin{bmatrix} V_u & 0 \\ 0 & V_y \end{bmatrix}$$

Then, the following algorithm can be used to find a q-MC.

The qMC algorithm: Assume the disturbance covariance W of w is given.

Step 1. Compute the data set $\boldsymbol{D}_q$ from the closed loop response ζ_k with respect to ν_k $(k = 0, 1, \dots, l)$.

Step 2. Compute the data matrix

$$\mathsf{D} = \mathsf{R}_{\zeta\zeta} - \mathsf{R}_{\zeta\nu}\mathsf{V}^{-1}\hat{\mathsf{V}}\mathsf{V}^{-\mathrm{T}}\mathsf{R}^{\mathrm{T}}_{\zeta\nu}$$

where

$$\mathsf{V} = I \otimes V$$

and

$$\hat{\mathsf{V}} = I \otimes (\Omega \times V)$$

If $\mathsf{D} \geq 0$, find a full rank matrix factor O_q

$$\mathsf{D} = \mathsf{O}_q\mathsf{O}_q^{\mathrm{T}}$$

Go to Step 3. If $\mathsf{D} \not\geq 0$, there is no linear model matching the data set $\boldsymbol{D}_q$. In this case an approximation is possible. Compute the spectral decomposition of D

$$\mathsf{D} = E_+\Lambda_+E_+^{\mathrm{T}} + E_-\Lambda_-E_-^{\mathrm{T}}$$

where Λ_+ contains the positive eigenvalues, Λ_- contains the non-positive eigenvalues, and E_+ and E_- are the corresponding eigenvectors. Now we replace data matrix D by $\hat{\mathsf{D}}$ with the approximation

$$\hat{\mathsf{D}} \triangleq E_+\Lambda_+E_+^{\mathrm{T}} \geq 0$$

Denote the corresponding full rank matrix factor of $\hat{\mathsf{D}}$ as O_q and go to the next step.

Step 3. Compute

$$\mathsf{M}_q = [R^{\mathrm{T}}_{\zeta\nu_0} \quad R^{\mathrm{T}}_{\zeta\nu_1} \quad \cdots \quad R^{\mathrm{T}}_{\zeta\nu_q}]^{\mathrm{T}} V^{-1}$$

$$\hat{\mathsf{O}}_{q-1} = [I_{n_y(q-1)} \quad 0]\mathsf{O}_q$$

Let $(\hat{A}, \hat{B}, \hat{C}, \hat{D})$ be the system matrix, then the following matrix equation holds

$$\begin{bmatrix} \hat{D} & \hat{C} \\ \hat{B}\hat{V}^{1/2} & \hat{A} \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & \hat{\mathsf{O}}^{+}_{q-1} \end{bmatrix} [\mathsf{M}_q \quad \mathsf{O}_q]$$

Notice that the above identification is independent of the *control-energy-sensitivity* weight Q. The following studies how to incorporate this Q information in the closed loop identification.

Let

$$Q = \mathrm{diag}\,(Q_1, Q_2, \dots, Q_n) \in \boldsymbol{P}$$

be the Kuhn–Tucker weight obtained from the OCC control design. Assume the closed-loop identification I/O data generated from system $T(G, K)$ in figure 1 are

$$\nu \triangleq [\nu_0, \nu_1, \dots, \nu_{n_d}]$$

$$\zeta \triangleq [\zeta_0, \zeta_1, \dots, \zeta_{n_d}] = \left[\begin{bmatrix} u_0 \\ y_0 \end{bmatrix}, \begin{bmatrix} u_1 \\ y_1 \end{bmatrix}, \dots, \begin{bmatrix} u_{n_d} \\ y_{n_d} \end{bmatrix}\right]$$

for a given data length n_d that is sufficiently large.

The weighted output data $\tilde{y}$ of y can be computed as

$$\tilde{y} = \sqrt{Q}y = [\sqrt{Q}y_0, \sqrt{Q}y_1, \dots, \sqrt{Q}y_{n_d}]$$

Hence the weighted output data for the closed-loop identification is

$$\tilde{\zeta} = \left[\begin{bmatrix} u_0 \\ \sqrt{Q}y_0 \end{bmatrix}, \begin{bmatrix} u_1 \\ \sqrt{Q}y_1 \end{bmatrix}, \dots, \begin{bmatrix} u_{n_d} \\ \sqrt{Q}y_{n_d} \end{bmatrix}\right]$$

For the purpose of ordering, the scaled $Q\|Q\|^{-1}$ is used to replace Q, In this case, we denote

$$Q_a = \begin{bmatrix} \|Q\|I & 0 \\ 0 & Q \end{bmatrix} \|Q\|^{-1} \qquad (25)$$

Then the weighted output response data $\tilde{\zeta}$ satisfies

$$\tilde{\zeta} = \sqrt{Q_a}\zeta$$

The auto-correlation coefficients of $\tilde{\zeta}$ and the cross-correlation coefficients between $\tilde{\zeta}$ and ν can be computed as

$$R_{\tilde{\zeta}\tilde{\zeta}_i} = \lim_{k\to\infty} \mathbf{E}[\tilde{\zeta}_{k+i}\tilde{\zeta}_k^{\mathrm{T}}]$$
$$R_{\tilde{\zeta}\nu_i} = \lim_{k\to\infty} \mathbf{E}[\tilde{\zeta}_{k+i}\nu_k^{\mathrm{T}}]$$

Using those $R_{\tilde{\zeta}\tilde{\zeta}_i}$ and $R_{\tilde{\zeta}\nu_i}$ to construct the Toeplitz matrices like those in (24), denote them as $\mathsf{R}_{\tilde{\zeta}\tilde{\zeta}}$ and $\mathsf{R}_{\tilde{\zeta}\nu}$. Compute the data matrix

$$\tilde{\mathsf{D}} = \mathsf{R}_{\tilde{\zeta}\tilde{\zeta}} - \mathsf{R}_{\tilde{\zeta}\nu}\mathsf{V}^{-1}\hat{\mathsf{V}}\mathsf{V}^{-\mathrm{T}}\mathsf{R}_{\tilde{\zeta}\nu}^{\mathrm{T}}$$

Normally the qMC for the weighted system is obtained by deleting zero eigenvalues of $\tilde{\mathsf{D}}$, but due to computational errors, none of the eigenvalues are exactly zero. Hence, those eigenvalues of $\tilde{\mathsf{D}}$ which have the smallest values are deleted, and the resultant qMC is called a truncated qMC. The following theorem holds.

Theorem 4: *Assume the system quadrupole* $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ *is a truncated qMC generated from* $\tilde{\mathsf{D}}$.

(i) *If the weight Q is invertible, then the truncated qMC of the original unweighted system which accommodates the control-energy-sensitivity weight Q will have system matrices*

$$A = \tilde{A}, \quad B = \tilde{B}, \quad C = Q^{-1/2}\tilde{C}, \quad D = Q^{-1/2}\tilde{D}$$

(ii) *If the control-energy-sensitivity weight Q is not invertible, a truncated qMC for the original system can be obtained by using the qMC algorithm to the following truncated data matrix (without weighting)*

$$\bar{\mathsf{D}} = \bar{E}_+\bar{\Lambda}_+\bar{E}_+^{\mathrm{T}}$$

where

$$\bar{\Lambda}_+ = \mathrm{diag}\,(\bar{\sigma}_1, \bar{\sigma}_2, \ldots, \bar{\sigma}_r)$$
$$\bar{E} = [\bar{e}_1 \quad \bar{e}_2 \quad \cdots \bar{e}_r]$$

and $[\bar{\sigma}_1, \bar{\sigma}_2, \ldots, \bar{\sigma}_r]$ are the r eigenvalues of the unweighted data matrix D which corresponds to the r largest Q-weight model costs defined by

$$M_i = \sigma_i e_i^{\mathrm{T}} Q_a e_i, \quad i \in \{1, 2, \ldots, qn_y\}$$
$$Q_a = I \otimes Q_a$$

and the $\bar{e}_i$'s are the associated eigenvectors of $\bar{\sigma}_i$'s.

Proof: (i) is obvious. Now we consider the proof for (ii). Since the weighted data satisfy

$$R_{\tilde{\zeta}\tilde{\zeta}_i} = \sqrt{Q_a}R_{\zeta\zeta_i}\sqrt{Q_a}$$
$$R_{\tilde{\zeta}\nu_i} = \sqrt{Q_a}R_{\tilde{\zeta}\nu_i}$$

Consider

$$(I \otimes \sqrt{Q_a})(I \otimes \sqrt{Q_a}) = I \otimes \sqrt{Q_a}$$

Hence

$$\sqrt{Q_a} = \sqrt{I \otimes Q_a} = I \otimes Q_a$$

This implies the equivalences

$$\tilde{\mathsf{D}} = \sqrt{Q_a}\mathsf{R}_{\zeta\zeta}\sqrt{Q_a} - \sqrt{Q_a}\mathsf{R}_{\zeta\nu}\mathsf{V}^{-1}\hat{\mathsf{V}}\mathsf{V}^{-\mathrm{T}}\mathsf{R}_{\zeta\nu}^{\mathrm{T}}\sqrt{Q_a}$$
$$= \sqrt{Q_a}\mathsf{D}\sqrt{Q_a}$$

Let the eigenvalues of $\tilde{\mathsf{D}}$ be $\tilde{\sigma}_1, \tilde{\sigma}_2, \ldots, \tilde{\sigma}_{qn_y}$, then the last equality implies

$$\sum_{i=1}^{qn_y} \tilde{\sigma}_i = \mathrm{tr}\,(\mathsf{D}Q_a)$$

If the ith Q-weighted modal cost is denoted as

$$M_i = \sigma_i e_i^{\mathrm{T}} Q_a e_i$$

then

$$\sum_{i=1}^{qn_y} \tilde{\sigma}_i = \sum_{i=1}^{qn_y} M_i$$

Those eigenvalues of $\tilde{\mathsf{D}}$ which have the smallest M_i values are deleted because of the above equality. If we keep the r largest $M_1, M_2, \ldots, M_r$ in the truncation, then the r eigenvalues of D, corresponding to $M_1, M_2, \ldots, M_r$, will be the main contributor to a model which accommodates *control-energy-sensitivity* information. □

After using the qMC algorithm to find a model for the *closed-loop* system, the next task is to extract a plant model from this identified closed-loop model. Several ways have been proposed to do this. The method of Liu and Skelton (1993) simply subtracts from the closed-loop model the known controller dynamics. This leads to a state model of the order of the identified closed-loop system plus the order of the controller (that is, equal to the plant order plus twice the controller order). Hence, model reduction is required to reduce the augmented system to a minimal realization. The approach here yields a plant representation that is of the order of the identified closed-loop system (of order of the plant plus controller). This approach to extract the plant from the closed loop model was first proposed by van der Klauw *et al.* (1991), where both the plant and the controller are assumed to be linear systems. While this model is of lower order than the one in Liu and Skelton (1993), it must still be reduced to a minimal realization.

Consider the asymptotically stable closed-loop system $T(G,K)$ depicted in figure 1 which is obtained by using a linear controller K to stabilize the real plant G. Since $T(G,K)$ represents the real physical system, no mathematical model can be obtained to capture all properties of $T(G,K)$. However, it is possible to find a linear model to capture certain properties of $T(G,K)$. For example, for q small enough, it is possible to obtain a qMC for $T(G,K)$. Suppose $\hat{T}(G,K)$ is a qMC for the closed-loop system. We want to construct a linear state space model for the real plant G based on $\hat{T}(G,K)$. Since $\hat{T}(G,K)$ is a linear system of inter-connection depicted in figure 1, there must exist a linear system $\hat{G}$ and $\hat{K}$ such that the transfer function matrix of $\hat{T}(G,K)$ satisfies

$$\hat{T}(G,K)(s) = \begin{bmatrix} I & -\hat{K}(s) \\ -\hat{G}(s) & I \end{bmatrix}^{-1} \begin{bmatrix} I & \hat{K}(s) \\ 0 & 0 \end{bmatrix} \tag{26}$$

Assume $\hat{T}(G,K)$ is obtained by using the identification scheme described in the last section and its state space description is

$$\hat{T}(G,K) \overset{\text{SSR}}{=} \left[\begin{array}{c|c} \hat{A} & \hat{B} \\ \hline \hat{C} & \hat{D} \end{array}\right]$$

Partition $\hat{B}$, $\hat{C}$ and $\hat{D}$ according to the dimensions of v_u, v_y, u and z, rewrite $\hat{T}(G,K)$ as

$$\hat{T}(G,K) \overset{\text{SSR}}{=} \left[\begin{array}{c|cc} \hat{A} & \hat{B}_1 & \hat{B}_2 \\ \hline \hat{C}_1 & \hat{D}_{11} & \hat{D}_{12} \\ \hat{C}_2 & \hat{D}_{21} & \hat{D}_{22} \end{array}\right] \tag{27}$$

and compute the matrix partition

$$\begin{bmatrix} \hat{A}_p & \hat{B}_p \\ \hat{C}_p & \hat{D}_p \end{bmatrix} = \begin{bmatrix} \hat{D}_{21} & \hat{C}_2 \\ \hat{B}_1 & \hat{A} \end{bmatrix} \begin{bmatrix} \hat{D}_{11} & \hat{C}_1 \\ 0 & I \end{bmatrix}^{-1} \tag{28}$$

Then the linear system $\hat{G}$ satisfying (26) can be described by the state space form

$$\left.\begin{aligned} x_{k+1} &= \hat{A}_p x_k + \hat{B}_p(u_k + w_k) \\ y_k &= \hat{C}_p x_k + \hat{D}_p(u_k + w_k) + s_k \end{aligned}\right\} \tag{29}$$

In this paper, we take the linear system $\hat{G}$ described in (29) as our model for control design. Note that this model is extracted from closed-loop response data.

Remark: We are only interested in a plant model which is good for control design. Hence whether $\hat{G}$ is close to G is not our concern here. It is known that the open-loop error $\hat{G} - G$ and the closed-loop error $\hat{T}(G,K) - T(G,K)$ may be quite different from each other. Therefore the plant model $\hat{G}$ might be quite different from G.

4. Integrated process

Now let us consider integrating closed-loop modelling with OCC control design. In simple terms, the closed-loop modelling and the OCC algorithm are iteratively combined together through the CES generated from the last control design step. A structure control problem using a similar algorithm can be found in Lu and Skelton (1998). The convergence of this algorithm cannot be proven mathematically at this time. However, our numerical experiences with several real world applications show the algorithm does converge. Further work will be conducted to address the convergence issues of the current algorithm.

The problem can be stated as the following: given a stable system G, an OCC-feasible covariance bound

$$\bar{Z} = \text{block diag}\,(\bar{Z}_1, \bar{Z}_2, \ldots, Z_n)$$

find a low order controller, by integrating closed-loop modelling and OCC control design, such that:

(i) the actual closed-loop system is stable;

(ii) the actual closed-loop system has output covariance bounded by $\bar{Z}$.

The following provides a step-by-step summary of the integration.

Integrated closed-loop modelling and control algorithm:

Step 0. Let $\bar{Z}$ be the given covariance bound. Choose an integer q (number of Markov/covariance parameters to be matched); another integer n_d (length of the experimental data).

Step 1. Set $i = 0$. If an analytic description of the plant is available, set $\hat{G}_i$ as the description of this model and go to Step 2. Otherwise, proceed to identify a plant model $\hat{G}_i$ by the noisy qMC identification algorithm.

Step 2. Control Design. Do model reduction for $\hat{G}_i$ to obtain a lower order model $\hat{G}_{ir}$ which is used as the design model for OCC control design. $\hat{G}_{ir}$ is called the ith design model. Choose an OCC-feasible covariance bound $\bar{Z}_d$ for the design model $\hat{G}_{ir}$ using solely the information from $\hat{G}_{ir}$. If $\hat{G}_{ir}$ is stable, an example of $\bar{Z}_d$ is (see Theorem 1)

$$\bar{Z}_d = \kappa \underline{Z}_d + (1-\kappa) Z_{d_{ol}}$$

and if $\hat{G}_{ir}$ is unstable, an example of $\bar{Z}_d$ is

$$\bar{Z}_d = \underline{Z}_d / \kappa$$

for properly chosen κ satisfying $\kappa \in (0,1)$. For this OCC-feasible bound, a $\bar{Z}_d$-suboptimal controller can be obtained through the OCC Algorithm 1, and the $\bar{Z}_d$-optimal controller can be obtained through the iteration defined in Theorem 2. Denote the $\bar{Z}_d$-optimal controller as K^i_{opt}. Store the final Kuhn–Tucker weight as Q.

Step 3. Control evaluation. Evaluate the controller K^i_{opt} with the real plant through white noise excitation and compute the output covariance. If the closed-loop system is unstable, the design specification (the covariance bound $\bar{Z}_d$) in Step 2 is too tight and must be relaxed. Decrease κ and go to Step 2. Otherwise check whether

$$\mathbf{E}_\infty[zz^{\mathrm{T}}] < \bar{Z}$$

if this is true, then Stop; otherwise the design model must be updated and go to Step 4.

Step 4. Closed-loop modelling. Generate data $R_{\zeta\zeta_i}$ and $R_{\zeta\nu_i}$ for $i = 0, 1, \dots, q-1$ by using white noise experiments. Form the data matrix D

$$\mathsf{D} = \mathsf{R}_{\zeta\zeta} - \mathsf{R}_{\zeta\nu}\mathsf{V}^{-1}\hat{\mathsf{V}}\mathsf{V}^{-\mathrm{T}}\mathsf{R}^{\mathrm{T}}_{\zeta\nu}$$

The weighted data matrix can be expressed as

$$\tilde{\mathsf{D}} = \sqrt{Q_a}\,\mathsf{D}\sqrt{Q_a}$$

Delete negative eigenvalues of D to get

$$\hat{\mathsf{D}} = E_+ \Lambda_+ E_+$$

where

$$\Lambda_+ = \operatorname{diag}(\sigma_1, \sigma_2, \dots, \sigma_r)$$
$$E_+ = [e_1 \quad e_2 \quad \cdots \quad e_r]$$

with $\sigma_i \geq 0$, $i = 1, 2, \dots, r$. Compute the modal costs

$$M_i = \sigma_i e_i^{\mathrm{T}} Q_a e_i$$

for $i = 1, 2, \dots, r$, where Q_a is computed from (25) with Q generated from Step 2. Choose the eigenvalues of D that correspond to largest model costs and choose the model order based on those eigenvalues. Use the formula in the noisy qMC algorithm to obtain a qMC of the closed-loop system and denote it as $\hat{T}(G_i, K^i_{\text{opt}})$. No model reduction is needed at this stage, i.e. a high order model of the closed-loop system is acceptable.

Step 5. Plant model extraction. Set $i = i + 1$. Partition the system matrix of the identified closed loop system $\hat{T}(G, K^{i-1}_{\text{opt}})$ as in (27). Then the plant model $\hat{G}_i$ can be extracted from $\hat{T}(G, K^{i-1}_{\text{opt}})$ as in § 4, where the system matrix quadruple of $\hat{G}_i$ is of the form (28). Here the order of $\hat{G}_i$ is the same as the order of $\hat{T}(G, K^{i-1}_{\text{opt}})$. Go to Step 2.

5. Example

Certain tensegrity structure controls the attitude of a rigid mass at the top of the structure. This is a structure of order 36. There are three outputs to control: the x, y and z translations of the rigid top. For the purpose of this study the digital simulation on a Sun SPARC station 5 is the real plant G. The description of the real plant dynamics is known only to the digital simulation but is unknown for identification, control design and performance evaluation procedures. All three procedures are blind to knowledge of the plant. The actuator and sensor are numerically implemented in the Sun SPARC station 5 to control the real plant G.

The open-loop covariance, computed from the white noise time response of G is

$$\bar{Z} = \begin{bmatrix} 1.3855 & -0.0597 & 0.1946 \\ -0.0597 & 0.3431 & 0.0944 \\ 0.1946 & 0.0944 & 1.3752 \end{bmatrix}$$

The objective of the integrated procedure is to find a model-based controller to satisfy a specified output variance bound for the actual closed-loop system. The following performance bounds are chosen as the requirements

$$\bar{Z}_1 = 0.56, \quad \bar{Z}_2 = 0.15, \quad \bar{Z}_3 = 0.56$$

The constants used here are

$$q = 150, \quad n_d = 20\,000$$

The initial model was identified by applying a white noise excitation to G and using the qMC algorithm to obtain a model $\hat{G}_0$ of order 40. A model reduction method (the covariance equivalent realization method, see Skelton (1988) and King *et al.* (1988) for detail) was used to reduce the 40th order $\hat{G}_0$ to the 26th order $\hat{G}_{0r}$. An OCC controller K_0 for $\hat{G}_{0r}$ was designed for a chosen design model performance specification $\bar{Z}_d$, corresponding to $\kappa = 0.56$. By numerically implementing K_0 to control G and computing the closed-loop output variance, the closed-loop performance could be evaluated. In this first iteration, the closed-loop output variances did not satisfy performance specification for the real closed-loop system. See table 1. Model iteration was needed. We updated the design model at the 2nd iteration. The output variance performance was not satisfied. We continued updating the model. At the fourth iteration, a satisfactory performance was achieved. Hence we stopped at the 4th iteration. Table 1 summarizes the iteration information.

In table 1, the chosen order of the design model is dictated by large gaps in the singular values of the D_q matrix during model reduction. Note that the system performance improves with iteration, even though the design model is getting simpler. This is possible due to

Iteration index	1	2	3	4
Order of design model $\hat{G}_{ir}$	26	21	18	18
Closed-loop output variance	$\begin{bmatrix}0.773\\0.206\\0.764\end{bmatrix}$	$\begin{bmatrix}0.661\\0.183\\0.647\end{bmatrix}$	$\begin{bmatrix}0.560\\0.155\\0.531\end{bmatrix}$	$\begin{bmatrix}0.553\\0.148\\0.435\end{bmatrix}$
κ	0.562	0.642	0.733	0.837

Table 1. Information summary of weighted closed-loop identification and OCC iteration.

2nd iteration	1	3	4	5	6	2	8	7	12	15	9	10	14	11	16
3rd iteration	1	3	4	7	6	2	9	8	5	12	13	10	22	11	18
4th iteration	2	1	4	6	7	10	8	3	5	15	9	13	12	31	30

Table 2. Modal cost ordering for closed-loop system at different iterations.

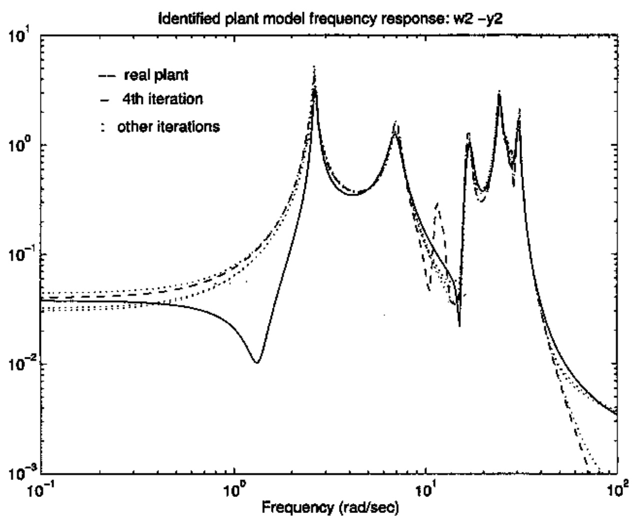

Figure 2. The Bode plots of the identified plant model transfer function from w_2 to y_2.

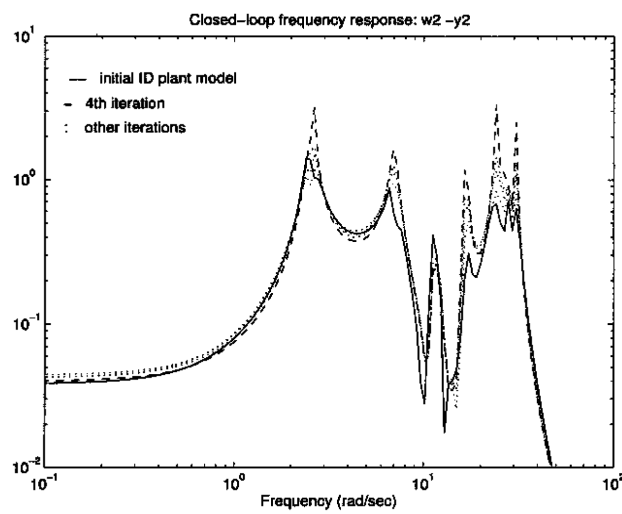

Figure 3. The Bode plots of the closed-loop transfer function w_2 to y_2.

the fact that the coordination of the design model and control dynamics increase the amount of essential information that is contained in the model.

Table 2 makes a comparison between the weighted and unweighted data processing. For example, the integers 1, 2, 3 etc. in this table refers to the modes of the singular values $\sigma_1, \sigma_2, \ldots, \sigma_n$ of the unweighted data matrix D_q. The positions in the table refer to the order of the model costs $M_i = \sigma_i e_i^{\mathrm{T}} Q e_i$ defined in Theorem 4.

Figure 2 compares the frequency response of all four design models during the iterative process. The frequency response of the actual plant is given by the dashed line. Note that a model that is good for closed-loop criterion (model 4) certainly would not be considered as a good model by an open-loop criterion (model 1). Figure 3 suggests that closed-loop performance is better using model 4 as the design model.

The closed-loop behaviour in figure 3 shows that the peak of all the modes are reduced, and the 5th and 7th peaks have the largest reduction. Figure 4 shows the

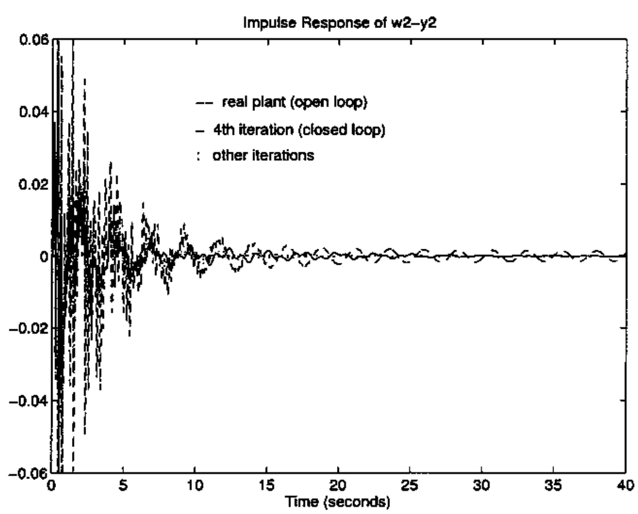

Figure 4. The impulse response of the closed-loop system in different iterations.

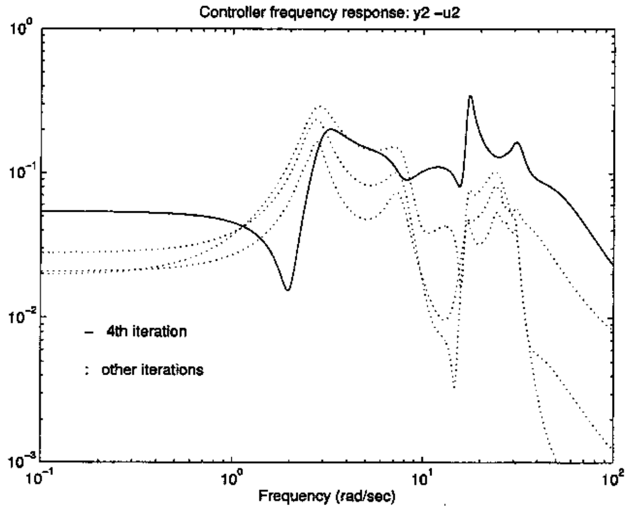

Figure 5. The Bode plots of the controller transfer function from y_2 to u_2.

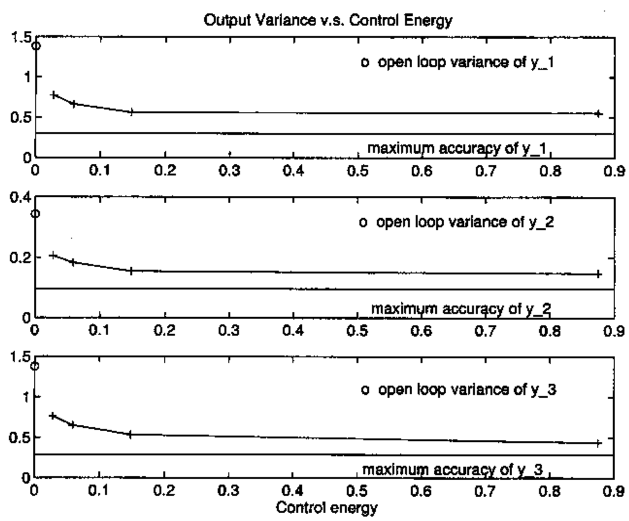

Figure 6. The closed-loop variance vs the control energy at each iteration.

impulse response of y_2. The control gains in channel $y_2 - u_2$ for all iterations are shown in figure 5. Figure 6 shows the variation of closed-loop output variance of different channels of the actual system with respect to the control energies.

6. Conclusion

The open-loop model error is not necessarily related to closed-loop errors. Since the ultimate goal of a control system is its closed-loop system behaviour, control design might find better controllers using a model reflecting feedback behaviours, which requires extracting a plant model from close-loop data. This philosophy allows the design model to be improved, as opposed to some existing approaches which simply specify the error bound for a nominal plant and to yield a robust controller. Although such approaches might achieve good robust stability, the robust performance focuses on minimizing an upper bound of the actual performance index, which could be very conservative. Our focus is close-loop performance index. The only kind of plant properties we consider are those that will allow successful closed-loop performance delivery.

The precise properties of a plant that are required to capture a successful model for control design remains an open question. This paper proposes an algorithm to receive closed-loop impulse or white noise response data together with a control energy sensitivity (CES) weight obtained in the last control design to produce a new design model. The new design model is used to derive a new controller and a new CES weight Q. This new controller is based upon a model that was extracted from the Q weighted closed-loop data and the q-Markov COVER (qMC) algorithm. The control design algorithm has three components: (i) an identification of the closed-loop system, (ii) a procedure to extract a plant model from the closed-loop system model, (iii) a control design procedure that guarantees (for the design model) an output covariance performance. Each of the three steps (i), (ii) and (iii) of the algorithm can guarantee the following: in (i) a linear model can be constructed that matches the auto-correlation and cross-correlation data from the real (non-linear) plant; in (ii) a model is constructed that (when driven by the known controllers) matches the identified closed-loop model; in (iii) the controller guarantees (for the design model) minimum control energy and the output covariance bound.

The example studied here demonstrates that an 18th order model can be used to design a controller for a 36th order plant. The controller that achieves a satisfactory output covariance constraint was found after four iterations. This model gives the BEST closed-loop performance and one of the WORST open-loop errors with respect to the actual plant. Thus, the example demonstrates the theme of the paper: models derived from closed-loop information can be quite different from models that would be considered good by an open-loop criterion.

The work reported here is along the line of the study involving using system methodologies to design integrated control systems as in Lu and Skelton (1999, 2000 b), where Lu and Skelton (1999) studied integration between the instrumentation and the control design using the finite signal-to-noise model considered in Lu and Skelton (2000 a), and Lu and Skelton (2000 b) studied integration between the structure design and the control design. The current paper together with Lu and Skelton (1999, 2000 b) provide small steps towards solving problems faced by rather complicated control system integrations.

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