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Integrating structure and control design to achieve mixed H_2/H_∞ performance

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Present technology in structure design (smart structures, civil structures and aerospace structures) includes the use of feedback control. While retrofitting such active elements can be useful in existing structures, future designs will require something more than retrofitting technology. Future technology will certainly require a more systematic integration of the design of a structure and its active elements. This paper provides a step in that direction. We seek to integrate the design of the structure with its active elements to achieve mixed H_2/H_∞ performance for the controlled structural system (closed loop system). More specifically, the approach presented here solves a mixed passive control (structure design) and active control (feedback control law design) problems with performance characterized by system norms such that H_∞ performance bounds are guaranteed with less active energy. This approach allows us to answer the question 'what is an optimal distribution of mass, stiffness, damping and control energy throughout a structure?' The main conclusion drew in this paper is that control design tools can be useful for structure design.

1. Introduction

The history of structure design divides into four eras. *The static era:* in the beginning structures were designed for their static responses. *The dynamic era:* in the second era, the dynamic response was considered (eigenvalues and mode shapes were restricted). *The hybrid era:* the third era involves structural response requirements so severe that they cannot be met with passive designs, hence feedback control is required and the existing control theory is used to find such feedback control law. In the static, dynamic and hybrid eras, the structures were (and are) designed by existing codes based upon classical structural design techniques. Feedback control in the hybrid area is added as an afterthought, after the structure is designed. *The system era:* the fourth era is the future, where new theoretic tools from control and system fields are used to design the structure and its feedback control if indeed control is needed. This new era will provide two advantages. Most researches either assume that feedback control is needed and proceed to optimize the feedback components, or that no feedback control is needed and proceed to optimize structural components. The new theory will determine whether control is needed and will optimize the structure and the controller jointly. The second advantage is the ability to modify standards to allow different criteria for design. For example, criteria that can limit the peak response in the presence of uncertain disturbances of a certain class (a known energy bound, etc.).

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It is a well-known fact that structure and control design problems are not independent, see Skelton (1989). Most often the two designs are competing with each other, and much control effort is wasted in combating dynamics of the structure. These dynamics could have been designed to cooperate more, without degrading other performance requirements (such as the static response). Therefore, for modern structures with severe constraints on the dynamic response, a theory for system design (i.e. integrating structure design with controller design) is required. An ideal approach is to simultaneously design the structure and the controller to optimize the structural dynamic response with respect to assumed disturbances and model uncertainties over feasible structures (for example, all feasible structures must have their masses and stiffnesses be bounded below in order to sustain static loads). In this paper, we treat the model as known, however the achieved H_∞ performance for the closed loop system implies that the system will be robust to certain structure uncertainties. In a true robust control setting, the joint structure and control design will find optimal nominal structure parameters and the feedback control law.

Let K denote a controller and p denote a parameter vector of a structure. The mathematical model of the structure corresponding to the free parameter p is denoted by $G(p)$. Let $\boldsymbol{P}$ be the range of the structure parameter p. Without loss of generality, $\boldsymbol{P}$ can be described by

$$\boldsymbol{P} \triangleq \{[p_1 \ p_2 \ \cdots \ p_n]^{\mathrm{T}} : \ p_i^- \le p_i \le p_i^+, \ i = 1, 2, \ldots, n\}$$

All the feasible structures with respect to $\boldsymbol{P}$ can be expressed as

$$\mathcal{G} \triangleq \{G(p) : \ p \in \boldsymbol{P}\}.$$

Let $J(G(p), K)$ be a measure of the structural dynamic response. Then the aforementioned ideal approach solves the following optimization problem,

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we should call it the *integrated structure and control* (ISC) problem

$$\min_{G\in\mathcal{G},K} J(G,K) = \min_{p\in\boldsymbol{P},K} J(G(p),K) \quad (1)$$

That is, this approach integrates *passive control* and *active control* design, where *passive control* modifies structure parameters (i.e. structure redesign, to find the free parameter p) without injecting external control energy, while *active control* uses external control energy (i.e. control design, to find the control law K for fixing structure parameter $p \in \boldsymbol{P}$). However there is no computationally tractable approach to solve (1) due to the complex and non-convex nature of optimization (1), see Onoda and Haftka (1987), Salama *et al.* (1988) and Ghandhi (1989). Notice that for fixed control K, the following optimization problem

$$\min_{p\in\boldsymbol{P}} J(G(p),K) \quad (2)$$

is a non-linear optimization problem, which is also a hard problem and there is no computationally tractable algorithm to solve it. Therefore the iteration between the regular control design (fix parameter p) and (2) would not be able to even find a locally optimal solution for (1).

In order to overcome the computational difficulty, Skelton and Kim (1992) and Grigoriadis *et al.* (1996) propose adding a constraint on the closed loop system in optimization (1). This added constraint changes (1) to a different but convex optimization, which has globally optimal solution. For a given $p \in \boldsymbol{P}$ and a controller K, if the closed loop system of using K to control $G(p)$ is

$$\dot{x}_{cl} = A_{cl}(p,K)x_{cl} + B_{cl}(p,K)w$$
$$z = C_{cl}(p,K)x_{cl} + D_{cl}(p,K)w$$

then the constraint used in Skelton and Kim (1992) and Grigoriadis *et al.* (1996) can be expressed as

> Modal equivalence: making the A_{cl} matrix of the closed loop system be equivalent before and after the optimization. That is
>
> $$A_{cl}(p,K) = A_{cl}(p^0,K^0)$$
>
> where p^0 and K^0 are the structure parameter and the controller before the optimization, and $A_{cl}(p^0,K^0)$ is the closed loop system matrix before the optimization.

More specifically, the problem solved in Skelton and Kim (1992) and Grigoriadis *et al.* (1996) can be described as

$$\min_{p\in\boldsymbol{P},K} \{\mathbf{E}_\infty[u^{\mathrm{T}}Ru] : \mathbf{E}_\infty[z_i^{\mathrm{T}}Q_iz_i] \le \gamma_i,\ i = 1,2,\ldots,m;$$
$$A_{cl}(p,K) = A_{cl}(p^0,K^0)\} \quad (3)$$

where $\mathbf{E}_\infty[\cdot]$ denotes the steady state expectation operator for stochastic processes, u is the control variable, γ_is reflect the performance level of the output variance. There are ways other than variances to properly characterize structure dynamic responses. For example, H_2 or H_∞ system norm can be used to characterize the structure dynamic responses. Good H_∞ performance implies either good robustness or good vibration attenuation with respect to all possible finite-energy disturbances, while H_2 performance characterizes the size of the response (energy) with respect to impulse or white noise disturbances. This motivates that we did not limit ourselves to variance performance requirement. As another extension, *modal equivalence* is the so called *system equivalence*. This equivalence will be denoted as

$$T_\infty(G(p),K) \in \boldsymbol{S}(p^0,K^0)$$

where $T_\infty(G(p),K)$ is some transfer matrix of the closed loop system. One of the properties of the above constraint is that the H_∞ norm of $T_\infty(G(p),K)$ is preserved, i.e. we have

$$\|T_\infty(G(p),K)\|_\infty = \|T_\infty(G(p^0),K^0)\|_\infty,$$
$$\forall T_\infty(G(p),K) \in \boldsymbol{S}(p^0,K^0) \quad (4)$$

Instead of limiting the output variance as in (3), mixed H_2/H_∞ specifications are considered, i.e. we would like to solve

$$\min_{p\in\boldsymbol{P},K} \{\|T_2(G(p),K)\|_2 : T_\infty(G(p),K) \in \boldsymbol{S}(p^0,K^0)\} \quad (5)$$

Due to (4), equation (5) automatically achieve the H_∞ norm constraints. That is if $\|T_\infty(G(p^0),K^0)\|_\infty \le \gamma_\infty$, then we must have $\|T_\infty(G(p),K)\|_\infty \le \gamma_\infty$. Because of this, we should call (5) the problem of *constrained optimal mixed passive and active control (COMPAC) with mixed H_2/H_∞ performance.* Another extension to Grigoriadis *et al.* (1996) is to consider more general structures which cover a large class of systems having plant disturbances/sensor noises, and the performance variables include accelerations of the structure. Many vibration control and vibration isolation problems can be cast into this framework.

The *integrated structure and control* (ISC) is achieved by iterating between the COMPAC problem and a mixed H_2/H_∞ control problem. The solution is a locally optimal one. For stable open loop systems, this approach always yields small control energy for sufficiently large performance level γ. If the control energy is zero, then the design is completely passive. The design always yields non-zero control energy if the performance level γ_is are small enough. In this case the selected performance cannot be achieved with any passive design. Hence the ISC design studied can address the following issues:

(1) Is active control necessary for a given mixed H_2/H_∞ performance requirement?

(2) If the answer is **yes**, provide a mixed passive and active solution to achieve this mixed H_2/H_∞ performance;

(3) If the answer to (i) is **no**, provide a passive solution, such that the mixed H_2/H_∞ performance is satisfied.

This paper is organized as the follows. Section 2 gives a brief discussion about structure system description. In § 3, a brief discussion about the mixed H_2/H_∞ control problem is given. Section 4 presents the solutions for optimal mixed passive and active control over system equivalent class. An iterative algorithm is given in § 5. An example is included in § 6.

The following notation will be used in this paper. $(\cdot)^{\mathrm{T}}$, $(\cdot)^{+}$ and $(\cdot)^{-1}$ denote the transpose, Moore–Penrose generalized inverse and inverse of a matrix. A positive definite matrix X is denoted as $X > 0$. A n by n unit matrix is denoted as $I_{n\times n}$. vec $(\cdot)$ operator stacks the columns of a matrix. $\otimes$ denotes the Kronecker product operation between two matrices. $\bar{\sigma}(\cdot)$ denotes the largest singular value of a matrix.

2. Structure system description

Consider a structure $G(p)$ with structural parameter $p \in \boldsymbol{P}$, whose state space description is given as

$$\left.\begin{aligned} E(p)(\dot{x} - B_1 w) &= A(p)x + B_2 u \\ z_\infty &= [x^{\mathrm{T}}F_1^{\mathrm{T}} \quad \dot{x}^{\mathrm{T}}F_2^{\mathrm{T}}]^{\mathrm{T}} \\ z_2 &= C_2 x + D_{22} u \\ y &= [x^{\mathrm{T}}M_1^{\mathrm{T}} \quad \dot{x}^{\mathrm{T}}M_2]^{\mathrm{T}} + Dv \end{aligned}\right\} \qquad (6)$$

where z_∞ is the output to which we impose an H_∞ performance requirement and z_2 is the output whose H_2 performance requirement is of interest. y is the measurement of the sensors available in the structure $G(p)$; $E(\cdot)$ and $A(\cdot)$ are linear functions of the forms

$$E(p) = E_0 + \sum_{i=1}^{n} p_i E_i, \qquad A(p) = A_0 + \sum_{i=1}^{n} p_i A_i$$

where $p \in \boldsymbol{P}$ and E_i, A_i are constant matrices for $i = 0, 1, \ldots, n$. We have the following assumptions for system (6) due to physical considerations

(A1) System (6) has independent measurements, i.e. M_1 and M_2 have full row rank.

(A2) System (6) has independent actuators, i.e. B_2 has full column rank.

(A3) E_0 is invertible.

Notice that a large class of lumped structural systems can be cast into the above form. Since in vibration control, the controlled variables are usually related to certain accelerations in the structure, hence the above setting has applications in structure vibration control. The following are two special cases of (6).

Example 1: Consider a mechanical system

$$M(p)(\ddot{q} + B_1 w) + D(p)\dot{q} + S(p)q = B_2 u$$

If some of the system absolute accelerations are included in both measurements and the performance variables, then this system can be cast into (6) with

$$E(p) = \begin{bmatrix} I & 0 \\ 0 & M(p) \end{bmatrix}, \qquad A(p) = \begin{bmatrix} 0 & I \\ -S(p) & -D(p) \end{bmatrix}$$

Example 2: The linear systems considered in Grigoriadis *et al.* (1996) can be recovered from (6) by setting $E(p) = I$, $F_2 = 0$, $D = 0$, $M_2 = 0$, $C_2 = 0$.

Our objective here is to find a parameter vector $p \in \boldsymbol{P}$ (passive control) and a possible feedback controller K such that an upper bound of the H_2 norm is minimized subject to an upper bound constraint on the closed loop system H_∞ norm. In the state feedback control case K represents

$$u = Hx \qquad (7)$$

and in the output feedback control case K represents

$$\left.\begin{aligned} \dot{x}_c &= A_c x_c + B_c y \\ u &= C_c x_c + D_c y \end{aligned}\right\} \qquad (8)$$

We have the following assumptions for the system (6) and the control (8)

(A4) If $D_c \neq 0$, then $M_2 = 0, D = 0$.

(A5) If $D \neq 0$, $M_2 \neq 0$, then $D_c = 0$.

Notice that the assumptions (A4) and (A5) guarantee the finiteness of the H_2 norm from $\mathbf{w}$ to z_2 of the system defined later in (9).

The *integrated structure and control system* uses structural parameter modification (passive control) and the active control (8) to control (6), which is of the form

$$\left.\begin{aligned} \dot{\mathbf{x}} &= \mathbf{A}(p, K)\mathbf{x} + \mathbf{B}(K)\mathbf{w} \\ z_\infty &= \mathbf{C}_\infty(p, K)\mathbf{x} + \mathbf{D}_\infty\mathbf{w} \\ z_2 &= \mathbf{C}_2(K)\mathbf{x} \end{aligned}\right\} \qquad (9)$$

where $\mathbf{w}$ denotes the disturbances applied to the closed loop system, $\mathbf{x}$ is the augmented state, i.e.

$$\mathbf{w} = \begin{bmatrix} w \\ v \end{bmatrix} \qquad \mathbf{x} = \begin{bmatrix} x \\ x_c \end{bmatrix}$$

If (A4) holds, then the system matrices of the *integrated structure and control system* (9) are

$$\mathbf{A}(p,K) = \begin{bmatrix} E(p)^{-1}(A(p)+B_2D_cM_1) & E(p)^{-1}B_2C_c \\ B_cM_1 & A_c \end{bmatrix}$$

$$\mathbf{B}(K) = \begin{bmatrix} B_1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\mathbf{C}_\infty(p,K) = \begin{bmatrix} F_1 & 0 \\ F_2E(p)^{-1}(A(p)+B_2D_cM_1) & F_2E(p)^{-1}B_2C_c \end{bmatrix}$$

$$\mathbf{D}_\infty = \begin{bmatrix} 0 & 0 \\ F_2\begin{bmatrix} B_1 \\ 0 \end{bmatrix} & 0 \end{bmatrix}$$

$$\mathbf{C}_2(K) = [C_2 + D_cM_1 \quad D_{22}C_c]$$

If (A5) holds, then the system matrices of the *integrated structure and control system* (9) can be written as

$$\mathbf{A}(p,K) = \begin{bmatrix} E(p)^{-1}A(p) & E(p)^{-1}B_2C_c \\ B_c\begin{bmatrix} M_1 \\ M_2E(p)^{-1}A(p) \end{bmatrix} & A_c + B_c\begin{bmatrix} 0 \\ M_2E(p)^{-1}B_2C_c \end{bmatrix} \end{bmatrix}$$

$$\mathbf{B}(K) = \begin{bmatrix} B_1 & 0 \\ B_c\begin{bmatrix} 0 \\ M_2B_1 \end{bmatrix} & B_cD \end{bmatrix}$$

$$\mathbf{C}_\infty(p,K) = \begin{bmatrix} F_1 & 0 \\ F_2E(p)^{-1}A(p) & F_2E(p)^{-1}B_2C_c \end{bmatrix}$$

$$\mathbf{D}_\infty = \begin{bmatrix} 0 & 0 \\ F_2B_1 & 0 \end{bmatrix}$$

$$\mathbf{C}_2(K) = [C_2 \quad D_{22}C_c]$$

For a given $p \in \boldsymbol{P}$ and a given controller K, denote the closed loop transfer function from $\mathbf{w}$ to z_2 as $T_2(G(p),K)$ and from $\mathbf{w}$ to z_∞ as $T_\infty(G(p),K)$.

In the following, $\|z(t)\|_2$ denotes the square root of the energy of a time-domain signal $z(t)$, i.e.

$$\|z(t)\|_2^2 = \int_0^\infty z^{\mathrm{T}}(t)z(t)\,\mathrm{d}t$$

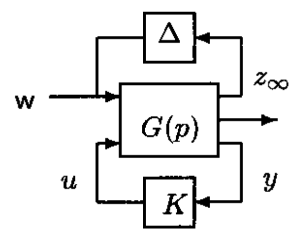
In the stochastic case, $\|z(t)\|_2^2$ denotes the variance of $z(t)$. Let $\mathcal{N}(0,W)$ denote a zero-mean white noise with intensity W. The H_2 norm of the transfer function $T_2(G(p),K)$ is defined as

$$\|T_2(G(p),K)\|_2 = \max\{\|z_2\|_2 : \ \mathbf{w} \in \mathcal{N}(0,W), \ W \leq I\}$$

The H_∞ norm of a linear time-invariant system can be computed from the frequency domain

$$\|T_\infty(G(p),K)\|_\infty = \sup_\omega \bar{\sigma}[T_\infty(G(p),K;j\omega)]$$

where $T_\infty(G(p),K;j\omega)$ denotes the frequency response of the transfer matrix of $T_\infty(G(p),K)$. Hence the H_∞ norm represents the peak magnitude (peak singular value) of the frequency response of a transfer matrix. This norm can characterize the robustness of a closed loop system with respect to norm bounded uncertainties, i.e. the system in the block diagram below will be stable for all uncertain dynamics Δ satisfying $\|\Delta\|_\infty \|T_\infty(G(p),K)\|_\infty < 1$.



System with uncertain dynamics

Another explanation of $\|T_\infty(G(p),K)\|_\infty$ is the square root of the energy amplification factor of the response z_∞ with respect to all possible inputs $\mathbf{w}$

$$\|T_\infty(G(p),K)\|_\infty^2 = \max_{\mathbf{w}}\left\{\frac{\|z_\infty\|_2^2}{\|\mathbf{w}\|_2^2} \ : \ \mathbf{w} \text{ has non-zero but finite energy}\right\}$$

3. Mixed H_2/H_∞ control

Consider the integrated structure and control system (9). The H_∞ norm of $T_\infty(G(p),K)$ does not exceed γ_∞ if and only if there exists a $P = P^{\mathrm{T}} > 0$ such that (see Boyd *et al.* 1994 and Skelton *et. al* 1997)

$$\begin{bmatrix} \mathbf{A}(p,K)P + P\mathbf{A}^{\mathrm{T}}(p,K) & \mathbf{B}(K) & P\mathbf{C}_\infty^{\mathrm{T}}(p,K) \\ \mathbf{B}^{\mathrm{T}}(K) & -I & \mathbf{D}_\infty^{\mathrm{T}} \\ \mathbf{C}_\infty(p,K)P & \mathbf{D}_\infty & -\gamma_\infty^2 I \end{bmatrix} < 0 \tag{10}$$

The H_2 norm of $T_2(G(p),K)$ does not exceed $\gamma_2 > 0$ if and only if there exists a $Q = Q^{\mathrm{T}} > 0$ such that

$$\left.\begin{aligned} &\begin{bmatrix} \mathbf{A}(p,K)Q + Q\mathbf{A}^{\mathrm{T}}(p,K) & \mathbf{B}(K) \\ \mathbf{B}^{\mathrm{T}}(K) & -I \end{bmatrix} < 0 \\ &\operatorname{tr}[\mathbf{C}_2(K)Q\mathbf{C}_2^{\mathrm{T}}(K)] < \gamma_2^2 \end{aligned}\right\} \quad (11)$$

For a fixed parameter $p \in \boldsymbol{P}$, finding K to satisfy (10) or (11) can be solved by the well-known H_∞ control or H_2 control theory. However for a fixed parameter $p \in \boldsymbol{P}$, finding a K to simultaneously satisfy (10) and (11) is an open problem and it may be computationally intractable. For tractability of computation in the LMI framework, a single Lyapunov matrix $X \triangleq P = Q$ is sought in the above conditions (see Chilali and Gahinet 1996 and Gahinet *et al.* 1995 for detail). We call this matrix X the H_2/H_∞ *common Lyapunov matrix*. This simplification leads to a performance upper bound for both H_2 and H_∞ norms of the closed loop system (9) for a given $p \in \boldsymbol{P}$. Denote the corresponding upper bounds for $\|T_2(G(p),K)\|_2$ and $\|T_\infty(G(p),K)\|_\infty$ as

$$\overline{\|T_2(G(p),K)\|_2}, \qquad \overline{\|T_\infty(G(p),K)\|_\infty}$$

We consider the following well-studied control synthesis problem.

Mixed H_2/H_∞ control problem: *For a fixed passive parameter vector $p \in \boldsymbol{P}$, solve for an active controller K from the following optimization problem*

$$J_{ac}(p) = \min_K \{\overline{\|T_2(G(p),K)\|_2} : \overline{\|T_\infty(G(p),K)\|_\infty} \le \gamma_\infty\} \quad (12)$$

The solution of the above problem can be obtained by using the LMI control toolbox, see Gahinet *et al.* (1995), and the following theorem holds.

Theorem 1: *For a fixed $p \in \boldsymbol{P}$, the mixed H_2/H_∞ control problem can be transferred to a convex optimization problem. Hence the optimal value $J_{ac}(p)$ is the global minimum of optimization* (12).

Proof: See Chilali and Gahinet (1996).

The ISC problem with mixed H_2/H_∞ performance can be generated from the above mixed H_2/H_∞ control problem.

Integrated structure and control with mixed H_2/H_∞ performance: Simultaneously solve for a controller K and a $p \in \boldsymbol{P}$ from the following optimization problem

$$J_{isc} = \min_{p \in \boldsymbol{P}, K} \{\overline{\|T_2(G(p),K)\|_2} : \overline{\|T_\infty(G(p),K)\|_\infty} \le \gamma_\infty\} \quad (13)$$

As mentioned in the introduction, the above problem is not solvable. In the following, we consider adding constraint on problem (13).

COMPAC with mixed H_2/H_∞ performance: Simultaneously solve for a controller K and a structure parameter vector $p \in \boldsymbol{P}$ from the following optimization problem

$$J_{\text{compac}} = \min_{p \in \boldsymbol{P}, K} \{\overline{\|T_2(G(p),K)\|_2} \ : \ T_\infty(G(p),K) \in \boldsymbol{S}(G(p^0),K^0)\}$$

for given structure parameter $p^0 \in \boldsymbol{P}$ and the control K^0. Where $\|T_\infty(G(p^0),K^0)\|_\infty \le \gamma_\infty$. $\boldsymbol{S}(G(p^0),K^0)$ is the system equivalent class, which will be defined in the next section.

In the following section, we provide the solutions for this COMPAC problem.

4. System equivalent COMPAC with mixed H_2/H_∞ performance

For a given $p \in \boldsymbol{P}$ and a given stabilizing controller K, the transfer matrix from $\mathbf{w}$ to z_∞ of (9) can be expressed as the following state space realization

$$T_\infty(G(p),K) \overset{\text{ssr}}{=} \left[\begin{array}{c|c} \mathbf{A}(p,K) & \mathbf{B}(p,K) \\ \hline \mathbf{C}_\infty(K) & \mathbf{D}_\infty \end{array}\right] \quad (14)$$

meaning that

$$T_\infty(G(p),K) = \mathbf{D}_\infty + \mathbf{C}_\infty(K)(sI - \mathbf{A}(p,K))^{-1}\mathbf{B}(p,K)$$

For two given pairs of $(\tilde{p},\tilde{K})$ and (p,K), $T_\infty(G(\tilde{p}),\tilde{K})$ is said to be *system equivalent* to $T_\infty(G(p),K)$ if all the involved system matrices are equal, i.e.

$$\mathbf{A}(\tilde{K},\tilde{p}) = \mathbf{A}(p,K) \quad (15)$$

$$\mathbf{B}(\tilde{K}) = \mathbf{B}(K) \quad (16)$$

$$\mathbf{C}_\infty(\tilde{K},\tilde{p}) = \mathbf{C}_\infty(p,K) \quad (17)$$

For a given controller K and a $p \in \boldsymbol{P}$, denote the whole class of *system equivalent* systems as $\boldsymbol{S}(p,K)$. In the following, without loss of generality, we consider the system equivalent class for $p = 0$, i.e. $\boldsymbol{S}(0,K)$. The following lemma provides the structure of all the transfer matrices in $\boldsymbol{S}(0,K)$.

Lemma 1: *Let K be given. Consider any given pair of structure parameter vector $p \in \boldsymbol{P}$ and feedback control law $\tilde{K}$. The following are true.*

(i) *If* (A4) *holds, then $T_\infty(G(p),\tilde{K}) \in \boldsymbol{S}(0,K)$ iff*

$$\tilde{A}_c = A_c, \qquad \tilde{B}_c = B_c$$

$$B_2\tilde{C}_c = B_2C_c + \sum_{i=1}^{n} E_iE_0^{-1}B_2C_cp_i \tag{18}$$

$$B_2\tilde{D}_cM_1 = B_2D_cM_1 + \sum_{i=1}^{n}[E_iE_0^{-1}(A_0 + B_2D_cM_1) - A_i]p_i \tag{19}$$

(ii) *If (A5) holds, then $T_\infty(G(p), \tilde{K}) \in \boldsymbol{S}(0, K)$ if and only if*

$$B_2\tilde{C}_c = B_2C_c + \sum_{i=1}^{n} E_iE_0^{-1}B_2C_cp_i$$

$$\sum_{i=1}^{n}[E_iE_0^{-1}A_0 - A_i]p_i = 0$$

and there exists a matrix Z such that

$$[\tilde{A}_c \quad \tilde{B}_c] = [A_c \quad B_c] + Z(YY^+ - I)$$

where

$$Y = \left[\begin{bmatrix} I \\ 0 \\ M_2E_0^{-1}B_2C_c \end{bmatrix} \begin{bmatrix} 0 \\ M_1 \\ M_2E_0^{-1}A_0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ M_2B_1 \end{bmatrix} \begin{bmatrix} 0 \\ D \end{bmatrix}\right]$$

(iii) *In the state feedback control (7) case, let the state feedback control gains matrix be $\tilde{H}$ and $\tilde{H}$. Then $T_\infty(G(p), \tilde{H}) \in \boldsymbol{S}(0, H)$ iff*

$$B_2\tilde{H} = B_2H + \sum_{i=1}^{n}[E_iE_0^{-1}(A_0 + B_2H) - A_i]p_i$$

Proof: See appendix B. □

Remark 1: The free matrix Z in (ii) of Lemma 1 does not change the closed loop system. Hence for simplicity, we could set $Z = 0$. Otherwise, Z could be arbitrarily chosen. The COMPAC with constraint limited to system equivalent class $\boldsymbol{S}(0, K)$, and achieving mixed H_2/H_∞ performance is to solve for $(\tilde{K}, p)$ from the following optimization problem

$$J_{\text{compac}} = \min\{\overline{\|T_2(G(p), \tilde{K})\|_2} : T_\infty(G(p), \tilde{K}) \in \boldsymbol{S}(0, K)\} \tag{20}$$

The following theorem provides the solution for this optimization.

Theorem 2: *For a given controller K and a H_∞ performance level $\gamma_\infty \geq \|T_\infty(G(0), K)\|_\infty$, let X_0 be the H_2/H_∞ common Lyapunov matrix defined in §3 for the closed loop system obtained by using K to control $G(0)$. Then the optimal solution for (20) is a convex, constrained quadratic optimization problem with respect to the plant parameters $p \in \boldsymbol{P}$, and can be reduced specifically to solving the following optimization problem*

$$J_{\text{compac}} = \min a(p)^{\mathrm{T}}(X_0 \otimes I)a(p) \quad \text{subject to} \quad p \in \boldsymbol{P}, \quad \hat{N}p = 0 \tag{21}$$

where

$$a(p) = \mathrm{vec}(W_0) + \hat{W}p$$

$$W = [\mathrm{vec}(W_1) \quad \mathrm{vec}(W_2) \quad \cdots \quad \mathrm{vec}(W_n)]$$

$$\hat{N} = [\mathrm{vec}(N_1) \quad \mathrm{vec}(N_2) \quad \cdots \quad \mathrm{vec}(N_n)]$$

If we denote

$$U_i = E_iE_0^{-1}B_2C_c$$

$W_i, N_i (i = 1, 2, \ldots, n)$ can be computed in the following for different cases

(i) *If* (A4) *holds, then*

$$W_0 = [C_2 + D_{22}D_cM_1 \quad D_{22}C_c]$$

$$W_i = [D_{22}B_2^+V_iM_1^+M_1D_{22}B_2^+U_i]$$

$$N_i = \begin{bmatrix} (I - B_2B_2^+)U_i \\ B_2B_2^+V_iM_1^+M_1 - V_i \end{bmatrix}$$

with $V_i = E_iE_0^{-1}(A_0 + B_2D_cM_1) - A_i$.

If p_{opt} is the optimal solution of (21), *then the optimal controller has the state space form*

$$K_{\text{opt}} \stackrel{\text{ssr}}{=} \left[\begin{array}{c|c} A_c & B_c \\ \hline C_{c_{\text{opt}}} = C_c + B_2^+\sum_{i=1}^{n} U_ip_{\text{opt}_i} & D_{c_{\text{opt}}} = D_c + B_2^+\sum_{i=1}^{n} V_iM_1^+p_i \end{array}\right]$$

(iii) *If* (A5) *holds, then*

$$W_0 = [C_2 \quad D_{22}C_c]$$

$$W_i = [0 \quad D_{22}B_2^+U_i]$$

$$N_i = \begin{bmatrix} (I - B_2B_2^+)U_i \\ E_iE_0^{-1}A_0 - A_i \end{bmatrix}.$$

If p_{opt} is the optimal solution of (21), *then the optimal controller has the state space form*

$$K_{\text{opt}} \stackrel{\text{ssr}}{=} \left[\begin{array}{c|c} \tilde{A}_{c_{\text{opt}}} & \tilde{B}_{c_{\text{opt}}} \\ \hline C_c + B_2^+\sum_{i=1}^{n} U_ip_{\text{opt}_i} & 0 \end{array}\right]$$

where

$$[\tilde{A}_{c_{\text{opt}}} \quad \tilde{B}_{c_{\text{opt}}}] = [A_c \quad B_c] + Z(YY^+ - I)$$

for some Z, where Y is defined in (ii) *of Lemma* 1.

(iii) *If the controller is state feedback and H is the given state feedback control gain matrix, then*

$$W_0 = H$$

$$W_i = E_i E_0^{-1}(A_0 + B_2 H) - A_i$$

$$N_i = (I - B_2 B_2^{+}) W_i$$

If p_{opt} is the optimal solution of (21), *then the optimal state feedback control gain matrix is*

$$H_{\text{opt}} = H + B_2^{+} \sum_{i=1}^{n} W_i p_{\text{opt}_i}$$

Proof: See Appendix B. □

Remark 2: The above optimization problem will have a non-zero solution for the passive parameter vector p if and only if

$$r = \text{rank } (\hat{N}) < n$$

In order to speed up the optimization, we decompose $\hat{N}$ as

$$\hat{N} = U \text{ diag}(\sigma_1^2, \ldots, \sigma_r^2, 0, \ldots, 0) V^{\text{T}}$$

and partition V according to the column dimension r

$$V = [V_r^{\text{T}} \quad *]$$

then

$$\hat{N} p = 0$$

is equivalent to

$$V_r p = 0$$

This observation leads to the truncation of the singular values of $\hat{N}$, i.e. for approximation, we might keep those columns of V which correspond to larger singular values. Also notice that the Matlab command *qp* can be directly used to solve the above constrained quadratic optimization problem.

5. Integrated structure and control design: an iteration scheme

Now we consider how to combine *system equivalent* COMPAC with mixed H_2/H_∞ performance and the mixed H_2/H_∞ control so as to achieve integrating structure and control design. The following algorithm iterates between those two problems. It can be summarized as follows. At each step, two tasks are performed. In the first task, the optimal performance is sought by solving the mixed H_2/H_∞ control problem for structure parameters p fixed at the previous step; in the second task, the system equivalent COMPAC with mixed H_2/H_∞ performance is conducted, and the optimal passive parameter $p \in \boldsymbol{P}$ and the feedback control parameters are simultaneously choosing to match the 4 state space system matrices of $T_\infty(G(p), K)$ with p and K being fixed in the previous step. Individually, any of those two tasks provides a global optimal solutions as mentioned in the last two sections, while the sequential combination of those two tasks only provides a locally optimal solution for ISC problem. However the convergence of this sequential combination is guaranteed.

Iterative algorithm:

Step 1. Set $k = 0$. Pick an initial parameter vector $p^k \in \boldsymbol{P}$ and formulate the state space system matrices as in (6).

Step 2. For the structure parameter p to be fixed in Step 1, find a controller to solve the following *mixed H_2/H_∞ control problem*

$$J_{ac}^k = \min_K \{\overline{\|T_2(G(p^k), K)\|_2} \ : \ \overline{\|T_\infty(G(p^k), K)\|_\infty} \le \gamma_\infty\}$$

and denote the global optimal controller as K^k. For this K^k, denote X^k as the H_2/H_∞ common Lyapunov matrix.

Step 3. For the given p^k, and the K^k, X^k obtained in step 2, solve the following *system equivalent COMPAC with mixed H_2/H_∞ performance*

$$J_{\text{compac}}^k = \min_{p,K} \{\overline{\|T_2(G(p), K)\|_2} : T_\infty(G(p), K) \in \boldsymbol{S}(p^k, K^k)\}$$

Notice that by rewriting

$$A(p) = A(p^k) + \sum_{i=1}^{n} A_i(p_i - p_i^k)$$

$$E(p) = E(p^k) + \sum_{i=1}^{n} E_i(p_i - p_i^k)$$

where $p \in \boldsymbol{P}$ and denoting $\Delta p = p - p^k$, then the above optimization is equivalent to

$$J_{\text{compac}}^k = \min \{\overline{\|T_2(G(p^k + \Delta p), K)\|_2} \\ : T_\infty(G(p^k + \Delta p), K) \in \boldsymbol{S}(0, K^k)\}$$

This problem is solved by Theorem 2. Denote the globally optimal passive parameter vector as $p^{k+1} = p^k + \Delta p$.

Step 4. if $|J_{ac}^k - J_{\text{compac}}^k| \le \epsilon$ (where ϵ is a given tolerance), then stop. Otherwise, set $k = k + 1$ and go to Step 2.

Theorem 3: *The above iterative algorithm guarantees converging to at least locally optimal solution.*

Proof: See Appendix B. □

6. Example

Using active controls in the protection of structures against environmental loads, such as strong earthquakes and high winds, has been a popular topic in the last two decades in structural control. In this section, integrated structure and control will be applied to the vibration control for a simulated building with seismic excitations. The example is tailored to simplify the discussion and to mainly illustrate the procedure outlined in the previous sections.

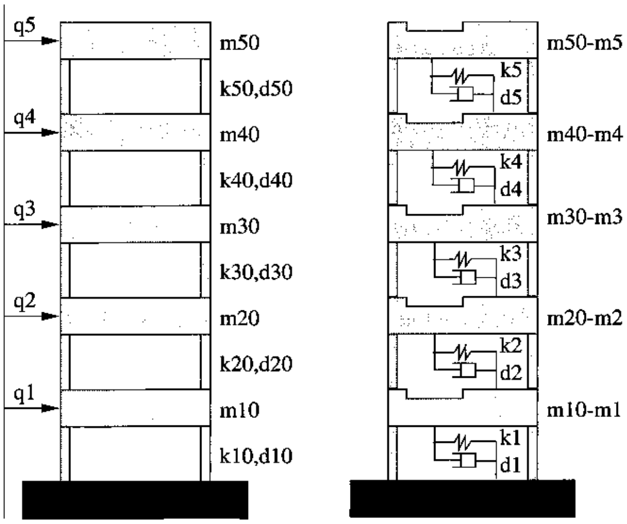
Consider the lateral motion of a 5-storey building subjected to a one-dimensional earthquake excitation $\ddot{q}_g$. The natural damping and stiffness might not be enough for the building to adequately suppress the vibrations caused by earthquakes. We are allowed to use integrated structure and control design to suppress the vibration of the building. The passive control can add damper and stiffness devices between floors and between the 1st floor and the ground, and remove certain amount of mass from each floor, see (*b*) of figure 1. Denote m_i, d_i, k_i as the mass, damping and stiffness coefficients of the *i*th floor of the building, and denote the adjustable parameters as

$$p = [k_5, \dots, k_1, d_5, \dots, d_1, m_5, \dots, m_1]^{\mathrm{T}}$$

then in (6) we have

$$E(p) = \begin{bmatrix} I & 0 \\ 0 & M(p) \end{bmatrix}, \qquad A(p) = \begin{bmatrix} 0 & I \\ -S(p) & -D(p) \end{bmatrix}$$

where the mass, damping, stiffness matrices can be written as linear functions of p in the forms



(a) Structure (b) Pasive control

Figure 1. A 5-storey building with structural modification (passive control).

$$M(p) = (m_5, m_4, \dots, , m_1)$$

$$D(p) = \begin{bmatrix} d_5 & -d_5 & & & \\ -d_5 & d_5 + d_4 & -d_4 & & \\ & -d_4 & d_4 + d_3 & -d_3 & \\ & & -d_3 & d_3 + d_2 & -d_2 \\ & & & -d_2 & d_2 + d_1 \end{bmatrix}$$

$$S(p) = \begin{bmatrix} k_5 & -k_5 & & & \\ -k_5 & k_5 + k_4 & -k_4 & & \\ & -k_4 & k_4 + k_3 & -k_3 & \\ & & -k_3 & k_3 + k_2 & -k_2 \\ & & & -k_2 & k_2 + k_1 \end{bmatrix}$$

Notice that q_i is the *i*th floor horizontal displacement relative to the ground. The active control devices might be chosen as tendon systems (figure 2 (*a*)) or as active brace systems (figure 2 (*b*)), but we consider the mathematically simplified case where the control forces are $u_1, u_2, \dots, u_5$ shown in figure 2, are distributed throughout the building.

Combining the passive modification (structure design) as in figure 1 and the active control as in figure 2, we obtain an *integrated structure and control system* which has the form (9).

In the following discussion, all the physical parameters are normalized to simple numbers such that the discussion emphasizes the mechanism of the method. The system state is

$$x(t) = [q^{\mathrm{T}}(t) \quad \dot{q}^{\mathrm{T}}(t)]^{\mathrm{T}}$$

where

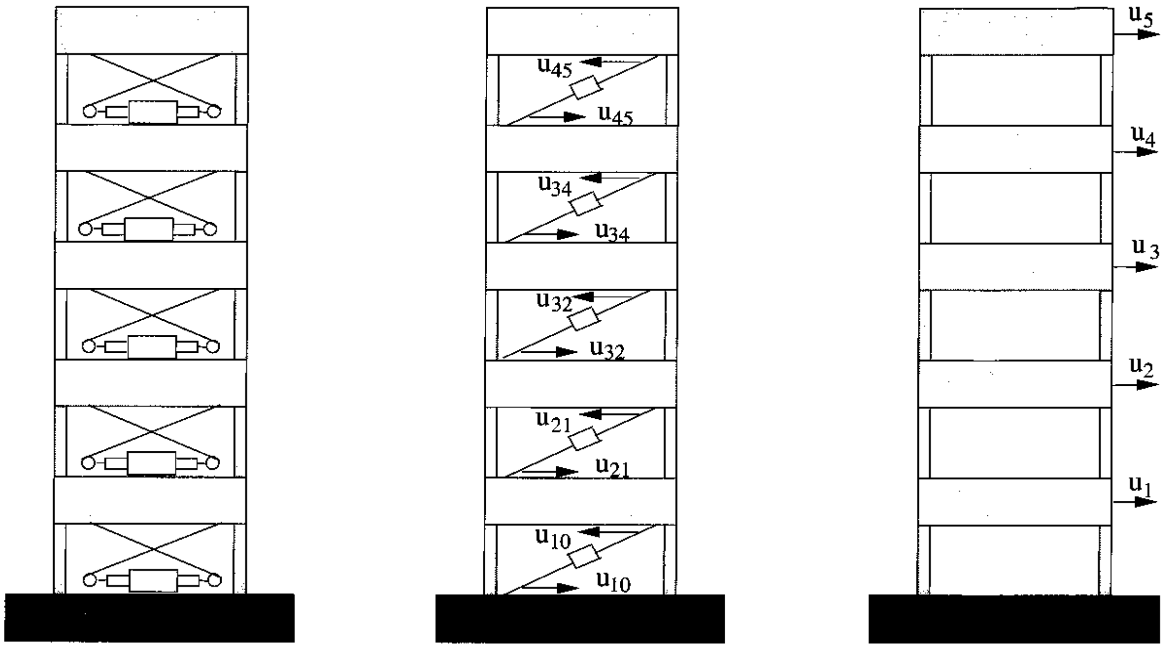
$$q(t) = [q_5(t) \quad q_4(t) \quad \cdots \quad q_1(t)]^{\mathrm{T}}$$

denotes the displacements relative to the ground of all the floors of the building and

$$B_1 = \begin{bmatrix} 0 \\ \mathbf{B}_1 \end{bmatrix} \qquad B_2 = \begin{bmatrix} 0 \\ \mathbf{B}_2 \end{bmatrix}$$

with

$$\mathbf{B}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \qquad \mathbf{B}_2 = I_{5\mathrm{x}5}$$



(a) Active tendon control (b) Active brace control (c) Mathematical Model

Figure 2. A 5-storey building with active control.

The unmodelled control-structure interaction, actuator/sensor dynamics and soil dynamics (which are all called the unmodelled dynamics) can severely limit both the performance and the robustness of the controlled structure. These unmodelled dynamics represent a good motivation for the robust feature of the H_∞ constraints in our design methodology. Using ISC with H_2/H_∞ performance, the active controller together with the passive solution will tolerate unmodelled dynamics of guaranteed size such that the performance of the controlled structure is not degraded when they are implemented in the real structures. The control objective is to limit the interstorey drift and the absolute acceleration of each floor, and at the same time to minimize the control energy. The interstory drift is defined as

$$z_{isd}(t) = \begin{bmatrix} 1 & -1 & & & \\ & 1 & -1 & & \\ & & 1 & -1 & \\ & & & 1 & -1 \end{bmatrix} q(t)$$

while the absolute acceleration vector of the building is

$$z_a(t) = \ddot{q}(t) + \mathbf{B}_1 \ddot{q}_g(t)$$

$z_{isd}(t)$ and $z_a(t)$ are chosen as the H_∞ performance variables, i.e.

$$z_\infty(t) = [z_{isd}^{\mathrm{T}}(t) \;\; z_a^{\mathrm{T}}(t)]^{\mathrm{T}}$$

and the H_2 performance variable is chosen as the active control $u(t)$. That is, we want to limit the dynamic response energy of z_{isd} and z_a with respect to any finite energy earthquake excitation $\ddot{q}_g(t)$, or, for an alternate interpretation, we seek a guaranteed robustness to tolerate the unmodelled dynamics from $\ddot{q}_g(t)$ to $z_\infty(t)$, with as low active control energy as possible, when the closed loop system is excited by white noise earthquake. Notice that if we impose a large performance bound on z_∞, the control energy might be zero. If we bound the z_∞ performance tightly enough, the control energy will not be zero. The sensor measurement here is taken as the displacement of each floor and the rates of the first, third and the fifth floors. Here we desire to know the optimal distributions of stiffness, damping and masses within the building for a given performance requirement.

Assume that the initial building in figure 1 is designed with lumped parameters

$$k_i^0 = 1, \quad d_i^0 = 0.02, \quad m_i^0 = 1, \quad i = 1, 2, \ldots, 5$$

Due to the static response requirements, the stiffness and mass must be bounded below. The damping must also be bounded above for cost and physical limitation considerations. Here as an example, we choose the following region for passive control

$$m_i \geq \tfrac{1}{2}(1.1 - 0.1i), \quad k_i \geq 0.5$$

$$0 \leq d_i \leq 0.2, \quad i = 1, 2, \ldots, 5$$

From static considerations, the mass of upper floors is allowed to be smaller than masses of the lower floors. We want to find the optimal structure parameters and the optimal active control $u(t)$ such that the H_∞ norm of the closed loop transfer function from $\ddot{q}_g(t)$ to $z_\infty(t)$ is bounded by a given number γ_∞ and the control energy is minimized.

We seek the optimal passive design. To do this, we iteratively increase γ_∞ until zero control energy is achieved. Such a solution is obtained for $\gamma_\infty = 1000$ (control energy $\|u(t)\|_2^2 < 10^{-4}$). The corresponding parameters of this passive design are

$$\begin{bmatrix} k_5 \\ k_4 \\ k_3 \\ k_2 \\ k_1 \end{bmatrix} = \begin{bmatrix} 0.5 \\ 0.5 \\ 0.5019 \\ 0.5084 \\ 0.5075 \end{bmatrix}, \quad \begin{bmatrix} d_5 \\ d_4 \\ d_3 \\ d_2 \\ d_1 \end{bmatrix} = \begin{bmatrix} 0.0086 \\ 0.0091 \\ 0.0091 \\ 0.0101 \\ 0.2000 \end{bmatrix}$$

$$\begin{bmatrix} m_5 \\ m_4 \\ m_3 \\ m_2 \\ m_1 \end{bmatrix} = \begin{bmatrix} 0.4312 \\ 0.4429 \\ 0.4546 \\ 0.4794 \\ 0.5043 \end{bmatrix}$$

Symbolically, the distributions of the passive parameters and the active control energies along the floors are shown in figure 3.

For $\gamma_\infty = 3$, the H_2/H_∞ OMPAC design provides an active controller with control energy $\|u(t)\|_2^2 = 0.1571$, and the parameters as

$$\begin{bmatrix} k_5 \\ k_4 \\ k_3 \\ k_2 \\ k_1 \end{bmatrix} = \begin{bmatrix} 0.6213 \\ 0.6673 \\ 0.7215 \\ 0.7918 \\ 0.5000 \end{bmatrix}, \quad \begin{bmatrix} d_5 \\ d_4 \\ d_3 \\ d_2 \\ d_1 \end{bmatrix} = \begin{bmatrix} 0.0060 \\ 0.0080 \\ 0.0080 \\ 0.0100 \\ 0.2000 \end{bmatrix}$$

$$\begin{bmatrix} m_5 \\ m_4 \\ m_3 \\ m_2 \\ m_1 \end{bmatrix} = \begin{bmatrix} 0.3 \\ 0.35 \\ 0.4 \\ 0.45 \\ 0.5 \end{bmatrix}$$

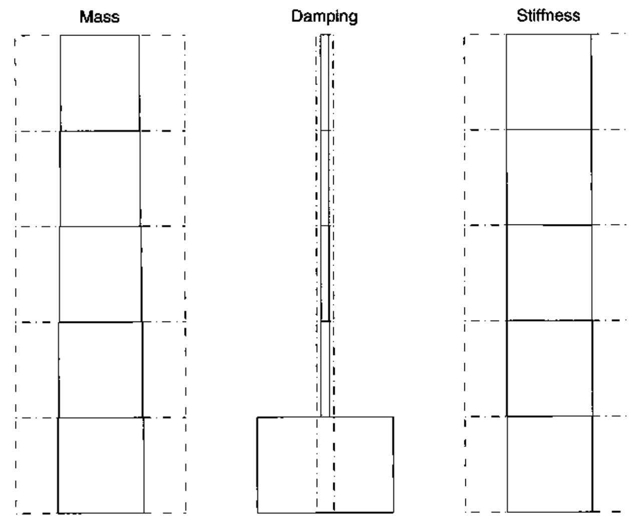


Figure 3. The distribution of mass, damping and stiffness along the floors. The initial building: dashed line; the optimal passive building: solid line. This case corresponds to the case of $\gamma_\infty \sim 1000$.

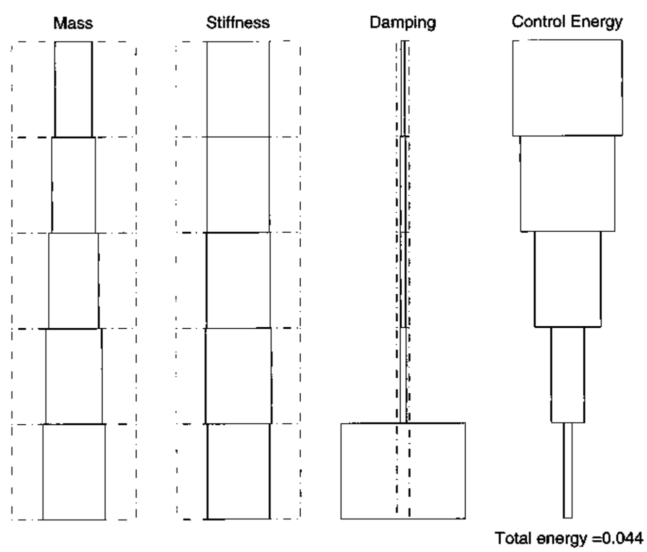


Figure 4. The distribution of mass, damping, stiffness and active control energy along the floors for the case of $\gamma_\infty = 10$. The initial building: dashed line; the building from the integrated structure and control design: solid line.

The distributions of the parameters are shown in figures 4, 5 and 7. Notice that for each design, the iteration presented in § 5 is conducted. The number of iterations required to achieve the stopping criterion (in Step 4) is usually affected by the tolerance ϵ. In all the iterations considered here, the tolerance is taken as $\epsilon = 10^{-2}$. Figure 6 shows that the algorithm converges after 20 iterations for $\gamma_\infty = 5$ case.

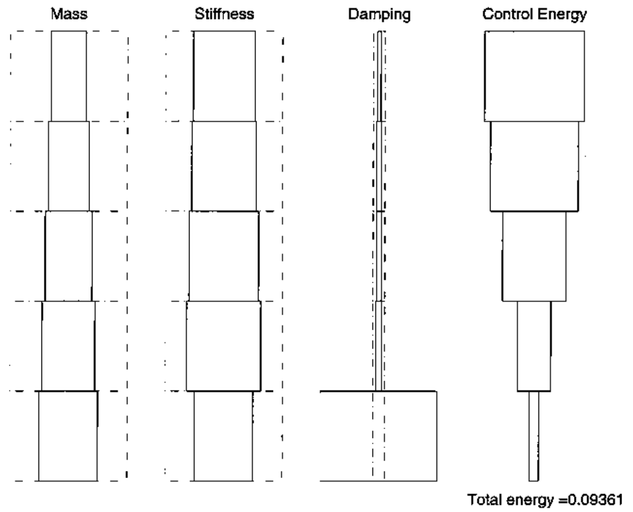


Figure 5. The distribution of mass, damping, stiffness and active control energy along the floors for the case of $\gamma_\infty = 5$. The initial building: dashed line; the building from the integrated structure and control design: solid line.

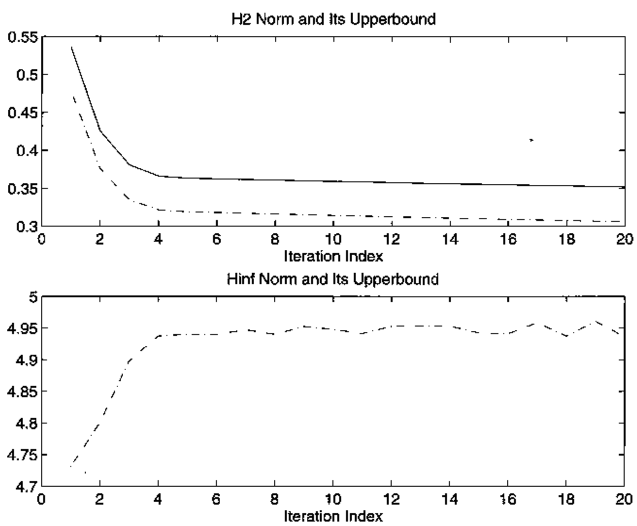


Figure 6. The iteration convergence study for the case of $\gamma_\infty = 5$. The upper plot: the H_2 norm (dashed line) and its upperbound (solid line) converge after 20 iterations for a tolerance of $\epsilon = 10^{-2}$. The lower plot: the H_∞ norm (dashed line).

We conclude this section with the following observation: the passive design occurs when the robustness upper bound is set to be $\gamma_\infty = 1000$; for stringent robustness requirement, the masses of all the floors reach the minimum physical bounds (our design used the minimal admissible mass); the bottom floor damping should be increased while its stiffness should be reduced as much as possible. This provides a mathematical justification for base isolation in civil structures. The

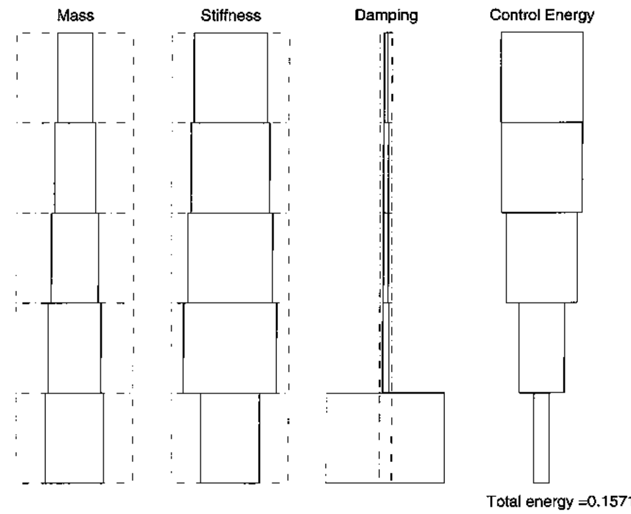


Figure 7. The distribution of mass, damping, stiffness and active control energy along the floors for the case of $\gamma_\infty = 3$. The initial building: dashed line; the building from the integrated structure and control design: solid line.

optimal damping and stiffness distributions are monotonically decreasing above the first floor.

7. Conclusion

This paper provides a technique to answer the questions: (i) 'for what constraints on the dynamic response is active control necessary?' (ii) 'what is the optimal stiffness, damping and mass distribution throughout a structure?', (iii) 'what is the optimal distribution of control energy throughout a structure?'. Such results depend strongly on the performance constraints imposed on the design. The method here integrates the structure design with the control design. The resultant system guarantees certain robustness (H_∞ norm is bounded) and minimum control energy (H_2 norm is optimized). If for the given robustness requirement, the minimum active control energy is zero, then a passive design to reach the guaranteed robustness is obtained. Due to the inevitable uncertainty such as unmodelled dynamics in real systems, this method can be of importance. The method is illustrated for a 5-storey building to obtain both passive and active designs.

Acknowledgments

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Appendix A. Preliminary results

The following theorem summarizes a well-known algebraic result, the proof of this theorem can be found in many algebra textbooks.

Theorem 4: *The linear matrix equation $AXB = C$ has a solution for X if and only if the following condition holds*

$$AA^+CB^+B = C$$

If the above condition holds, all the solutions of X can be parameterized as

$$X = A^+CB^+ + A^+AZBB^+ - Z$$

where Z is a free parameter of proper dimension. If further A has full column rank and B has full row rank, then we have a unique solution for X

$$X = A^+CB^+$$

Properties of the Kronecker product: For any matrices A, B, X with proper dimensions, the following hold

(i) $\mathrm{vec}(AXB) = (B^{\mathrm{T}} \otimes A)\,\mathrm{vec}(X)$

(ii) $\mathrm{tr}(AB) = \mathrm{vec}^{\mathrm{T}}(A^{\mathrm{T}})\,\mathrm{vec}(B)$.

Appendix B. Proof of theorems

Proof of Lemma 1: Let us first consider (1). If (A4) holds, the (2,1) element of $\mathbf{A}(\tilde{p}, K) = \mathbf{A}(0, K)$ satisfies

$$\tilde{B}_c M_1 = B_c M_1$$

Considering (A1) and Theorem 4 we must have $\tilde{B}_c = B_c$. The (1, 1) and (1, 2) elements of $\mathbf{A}(\tilde{p}, \tilde{K}) = \mathbf{A}(0, K)$ imply

$$\left.\begin{aligned} E^{-1}(p)(A(p) + B_2\tilde{D}_cM_1) &= E_0^{-1}(A_0 + B_2D_cM_1) \\ E^{-1}(p)B_2\tilde{D}_c &= E_0^{-1}B_2D_c \end{aligned}\right\} \tag{22}$$

(16) is automatically satisfied, and (17) implies the following due to the assumption (A3)

$$\left.\begin{aligned} F_2E^{-1}(p)(A(p) + B_2\tilde{D}_cM_1) &= F_2E_0^{-1}(A_0 + B_2D_cM_1) \\ F_2E^{-1}(p)B_2\tilde{C}_c &= F_2E_0^{-1}B_2C_c \end{aligned}\right\} \tag{23}$$

(22) and (23) imply

$$\left.\begin{aligned} &\begin{bmatrix} I \\ F_2 \end{bmatrix}[E^{-1}(p)(A(p) + B_2\tilde{D}_cM_1) - E_0^{-1}(A_0 + B_2D_cM_1)] = 0 \\ &\begin{bmatrix} I \\ F_2 \end{bmatrix}[E^{-1}(p)B_2\tilde{D}_c - E_0^{-1}B_2D_c] = 0 \end{aligned}\right\} \tag{24}$$

From Theorem 4, (24) is true iff (18) and (19) are true. Hence (i) is true.

Now we consider (ii). The (1, 1) and (1, 2) elements of $\mathbf{A}(\tilde{p}, \tilde{K}) = \mathbf{A}(0, K)$ imply

$$E(p)^{-1}A(p) = E_0^{-1}A_0, \qquad E(p)^{-1}B_2\tilde{C}_c = E_0^{-1}B_2C_c \tag{25}$$

which imply the first part of (ii). The (2, 1) and (2, 2) elements of $\mathbf{A}(\tilde{p}, \tilde{K}) = \mathbf{A}(0, K)$ together with (25) lead to

$$\left.\begin{aligned} &\tilde{B}_c\begin{bmatrix} M_1 \\ M_2E(p)^{-1}A(p) \end{bmatrix} = B_c\begin{bmatrix} M_1 \\ M_2E_0^{-1}A_0 \end{bmatrix} \\ &[\tilde{A}_c - A_c \quad \tilde{B}_c - B_c]\begin{bmatrix} I \\ 0 \\ M_2E_0^{-1}B_2C_c \end{bmatrix} = 0 \end{aligned}\right\} \tag{26}$$

The (2,1) element of $\mathbf{B}(\tilde{K}) = \mathbf{B}(K)$ implies

$$(\tilde{B}_c - B_c)\left[\begin{bmatrix} 0 \\ M_2B_1 \end{bmatrix} \quad D\right] = 0$$

Hence we have

$$[\tilde{A}_c - A_c \quad \tilde{B}_c - B_c]Y = 0 \tag{27}$$

From Theorem 4, there must exist a Z such that

$$[\tilde{A}_c - A_c \quad \tilde{B}_c - B_c] = Z(YY^+ - I)$$

Therefore (ii) is true.

Similar proof can be used for (iii). □

Proof of Theorem 2: We only consider the proof for (i), the other two cases can be addressed similarly. There exists a solution $\tilde{C}_c$ to solve (18) in Lemma 1 if and only if (see Theorem 4)

$$(I - B_2B_2^+)\sum_{i=1}^{n} U_ip_i = 0 \tag{28}$$

If this is true, the solution can be expressed as the following, due to the assumption (A2) and Theorem 4

$$\tilde{C}_c = C_c + B_2^+\sum_{i=1}^{n} U_ip_i \tag{29}$$

There exists a solution $\tilde{D}_c$ to solve (19) in Lemma 1 if and only if

$$B_2B_2^+\sum_{i=1}^{n} V_ip_iM_1^+M_1 = \sum_{i=1}^{n} V_ip_i \tag{30}$$

If the above is true, the solution for $\tilde{D}_c$ can be expressed as the following due to assumptions (A1), (A2) and Theorem 4

$$\tilde{D}_c = D_c + B_2^+\sum_{i=1}^{n}[E_iE_0^{-1}(A_0 + B_2D_cM_1) - A_i]M_1^+p_i \tag{31}$$

Consider

$$z_2 = C_2x + D_{22}u = (C_2 + D_{22}\tilde{D}_cM_1)x + D_{22}\tilde{C}_cx_c$$

for all $T_\infty(\tilde{K},p) \in \boldsymbol{S}(K,0)$, we have

$$\overline{\|T_2(\tilde{K},p)\|_2} = \mathrm{tr}([C_2 + D_{22}\tilde{D}_c M_1 \;\; D_{22}\tilde{C}_c]X_0 \times [C_2 + D_{22}\tilde{D}_c M_1 \;\; D_{22}\tilde{C}_c]^{\mathrm{T}})$$

where $\tilde{C}_c, \tilde{D}_c, p$ satisfy (28), (29), (30) and (31). Following these substitutions, we have

$$\overline{\|T_2(\tilde{K},p)\|_2} = \mathrm{tr}\left[\left(W_0 + \sum_{i=1}^{n} W_i p_i\right) \times X_0 \left(W_0 + \sum_{i=1}^{n} W_i p_i\right)^{\mathrm{T}}\right] = \mathrm{tr}\left[\left(W_0 + \sum_{i=1}^{n} W_i p_i\right)^{\mathrm{T}} \times \left(W_0 + \sum_{i=1}^{n} W_i p_i\right) X_0\right]$$

with p satisfying (28) and (30). Consider the properties of the Kronecker product (see the appendix), the above can be further written as

$$\overline{\|T_2(\tilde{K},p)\|_2} = \mathrm{vec}^{\mathrm{T}}\left(W_0 + \sum_{i=1}^{n} W_i p_i\right) \times \mathrm{vec}\left[\left(W_0 + \sum_{i=1}^{n} W_i p_i\right) X_0\right] = \mathrm{vec}^{\mathrm{T}}\left(W_0 + \sum_{i=1}^{n} W_i p_i\right)(X_0 \otimes I) \times \mathrm{vec}\left(W_0 + \sum_{i=1}^{n} W_i p_i\right)$$

Also notice that p must lie within $\boldsymbol{P}$ and satisfying the constraints (28) and (30), which can be expressed as

$$\sum_{i=1}^{n} N_i p_i = 0 \qquad (32)$$

Using the Kronecker product, (32) can be further expressed as

$$\hat{N}p = 0$$

Hence the optimal cost function J_{compac} can be computed in (21). □

Proof of Theorem 3: In the kth iteration, let K^k solve the optimization problem in Step 2. For this controller, denote

$$\overline{\|T_\infty(G(p^k), K^k)\|_\infty} = \gamma^k \leq \gamma_\infty$$

then the optimization for mixed H_2/H_∞ control in step 2 is equivalent to

$$J^k_{ac} = \min_K \overline{\|T_2(G(p^k), K)\|_2} \quad \text{subject to} \quad \overline{\|T_\infty(G(p^k), K)\|_\infty} = \gamma^k$$

and $J^k_{ac} = \overline{\|T_2(G(p^k), K^k)\|_2} \geq 0$.

Consider for all system equivalent $T_\infty(G(p), K) \in \boldsymbol{S}(p^k, K^k)$, we have

$$\overline{\|T_\infty(G(p), K)\|_\infty} = \gamma^k$$

Hence the optimization problem in Step 3 is equivalent to

$$J^k_{\mathrm{compac}} = \min_p \min_K \overline{\|T_2(G(p), K)\|_2} \quad \text{subject to} \quad T_\infty(G(p), K) \in \boldsymbol{S}(p^k, K^k)$$

Since the above problem is a convex optimization problem (see Theorem 2), hence J^k_{pac} is the global minimum. This implies

$$0 \leq J^k_{\mathrm{compac}} \leq J^k_{ac}$$

For the $(k+1)$th iteration, solve the following optimization problem

$$J^{k+1}_{ac} = \min_K \overline{\|T_2(G(p^{k+1}), K)\|_2} \quad \text{subject to} \quad \overline{\|T_\infty(G(p^{k+1}), K)\|_\infty} < \gamma_\infty$$

and denote the optimal solution as K^{k+1}. Since this optimization is convex, hence J^{k+1}_{ac} is the global minimum (see Theorem 1), therefore we must have

$$0 \leq J^{k+1}_{ac} \leq J^k_{\mathrm{compac}} \leq J^k_{ac}$$

This implies there exists a $\gamma_2 \geq 0$ such that

$$\gamma_2 = \min_k J^k_{ac} = \min_k J^k_{\mathrm{compac}} = \lim_{k\to\infty} J^k_{ac} = \lim_{k\to\infty} J^k_{pac}$$

or the iteration converges to γ_2. □

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