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L_2 and $L_2 - L_\infty$ Model Reduction via Linear Matrix Inequalities

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Abstract

Necessary and sufficient conditions are derived for the existence of a solution to the continuous-time and discrete-time suboptimal L_2 and $L_2 - L_\infty$ model reduction problems. These conditions are expressed in terms of Linear Matrix Inequalities (LMIs) and a coupling non-convex rank constraint set. In addition, explicit parametrizations of all reduced-order models that correspond to a feasible solution are presented in terms of contractive matrices.

1. Introduction

The model-order reduction problem consists of approximating a high-order system G by a lower-order system $\widehat{G}$ according to some given criterion. One measure of closeness of the full-order and the reduced-order models that has been examined extensively in the controls literature is the steady-state quadratically weighted output error of the two system models, when both models are subject to white noise inputs. A deterministic equivalent of this measure is the L_2 norm of the output error when both models are subject to unit impulses applied simultaneously to all inputs. The L_2 model reduction problem consists of finding a reduced-order model that minimizes this quadratic cost criterion.

First-order necessary conditions for L_2 model reduction were derived by Wilson [15], [16], in the form of two coupled Lyapunov equations of order equal to the summation of the full and the reduced-order models. An iterative scheme was proposed for solution, but with no guarantee of convergence. An optimal projection equation approach to the L_2 model reduction problem was developed in [7] where the necessary conditions for optimality were transformed to a pair of coupled modified Lyapunov equations of order equal to the order of the full-order model, with some additional rank constraints. A homotopy method to solve the optimal projection model reduction equations was proposed in [18]. Optimal projection equations for combined L_2/H_∞ model reduction were presented in [6].

The Hankel singular values of a system that are utilized in balance truncation may provide poor guidance for L_2 model reduction [11], [7]. In [11] the balanced gains of a system were introduced to describe the contribution of each state to the L_2 magnitude of the response, and an L_2 model truncation criterion was proposed based on the magnitude of these contributions. An L_2-based component cost decomposition for model reduction has been proposed in [12], [13].

The $L_2 - L_\infty$ model reduction problem consists of finding a reduced-order model that minimizes the L_∞ norm of the output error when both the full and the reduced-order models are subject to bounded L_2 norm excitations. For single-input-single-output systems this problem is equivalent to the L_2 model reduction [17]. $L_2 - L_\infty$ model reduction has been considered in [12] following a covariance control approach.

In the present work, necessary and sufficient conditions are derived for the solution of the continuous-time and discrete-time γ-suboptimal L_2 and $L_2 - L_\infty$ model reduction problems utilizing linear matrix inequalities (LMIs). Also, an explicit parametrization of all reduced-order models that correspond to a feasible solution is obtained. Numerical techniques based on alternating projections are proposed to solve these model reduction problem following the approach of [3], [4]. An LMI formulation of the H_∞ norm model reduction problem has been developed in [1] and [5]; see also [9], [10].

The notation used in this paper is as follows: $(\cdot)^T$ denotes the transpose and $(\cdot)^+$ denotes the Moore-Penrose generalized inverse of a matrix. The norm $\|\cdot\|$ denotes the maximum singular value norm of a matrix or the Euclidean norm of a vector. $\|\cdot\|_2$ stands for the L_2 norm and $\|\cdot\|_\infty$ stands for the L_∞ norm of a real function. The standard notation $>, \geq$ $(<, \leq)$ is used to denote the positive (negative) definite and semidefinite ordering of matrices.

2. Continuous-Time L_2 Model Reduction

Consider a stable nth-order strictly proper linear time-invariant system G with a state-space representation

$$\dot{x} = Ax + Bw \quad (1)$$

$$y = Cx \quad (2)$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$ and $C \in \mathbb{R}^{p\times n}$. The *optimal L_2 model reduction problem* is to find a stable $\hat{n}th$-order system $\widehat{G}$ with state-space representation

$$\dot{\hat{x}} = \widehat{A}\hat{x} + \widehat{B}w \quad (3)$$

$$\hat{y} = \widehat{C}\hat{x} \quad (4)$$

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where $\widehat{A} \in \mathbb{R}^{\widehat{n}\times\widehat{n}}$, $\widehat{B} \in \mathbb{R}^{\widehat{n}\times m}$, $\widehat{C} \in \mathbb{R}^{p\times\widehat{n}}$, and $\widehat{n} < n$, such that the L_2 error

$$\|y - \widehat{y}\|_2 = \left\{ \int_0^\infty \|y(t) - \widehat{y}(t)\|^2 dt \right\}^{1/2} \tag{5}$$

is minimized for all impulsive inputs w. That is, $w \in \mathcal{W}$ where the disturbance set $\mathcal{W}$ is defined as

$$\mathcal{W} = \{w_0\delta(t) \ : w_0 \in \mathbb{R}^m, \ \|w_0\| \le 1\}$$

where $\delta(\cdot)$ is the Dirac functional. The *γ-suboptimal L_2 model reduction problem* is to find $\widehat{G}$, if it exists, such that

$$\|y - \widehat{y}\|_2 < \gamma \tag{6}$$

where γ is a given positive scalar.

The following result provides necessary and sufficient conditions for the solution of the γ-suboptimal L_2 model reduction problem in terms of LMIs.

Theorem 1 *There exists an $\widehat{n}$th-order system $\widehat{G}$ to solve the γ-suboptimal L_2 model reduction problem if and only if there exist matrices $X > 0$ and $Y > 0$ such that the following conditions are satisfied*

$$X > \frac{1}{\gamma^2} BB^T \tag{7}$$

$$AX + XA^T < 0 \tag{8}$$

$$YA + A^TY + C^TC < 0 \tag{9}$$

$$\begin{bmatrix} X & I \\ I & Y \end{bmatrix} \ge 0 \tag{10}$$

and

$$\text{rank} \begin{bmatrix} X & I \\ I & Y \end{bmatrix} \le n + \widehat{n}\ . \tag{11}$$

Proof Consider the error system $\tilde{G} = G - \widehat{G}$ with state space representation

$$\begin{aligned} \dot{\tilde{x}} &= \tilde{A}\tilde{x} + \tilde{B}w \\ z &= \tilde{C}\tilde{x} \end{aligned}$$

where

$$\tilde{x} = \begin{bmatrix} x \\ \hat{x} \end{bmatrix},\ z = y - \hat{y}$$

$$\begin{aligned} \tilde{A} &= \bar{A} + \bar{B}\hat{G}_1\bar{M} \\ \tilde{B} &= \bar{D} + \bar{E}\hat{G}_2 \\ \tilde{C} &= \bar{C} + \bar{H}\hat{G}_1\bar{M} \end{aligned}$$

and

$$\hat{G}_1 = \begin{bmatrix} \hat{C} \\ \hat{A} \end{bmatrix},\ \hat{G}_2 = \hat{B},$$

$$\bar{A} = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}\ \overline{B} = \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix}\ \bar{D} = \begin{bmatrix} B \\ 0 \end{bmatrix}\ \overline{E} = \begin{bmatrix} 0 \\ I \end{bmatrix} \tag{12}$$

$$\overline{M} = \begin{bmatrix} 0 & I \end{bmatrix}\ \overline{H} = \begin{bmatrix} -I & 0 \end{bmatrix}\ \bar{C} = \begin{bmatrix} C & 0 \end{bmatrix}. \tag{13}$$

The γ-suboptimal L_2 model reduction problem has a solution if and only if there exists a matrix $\bar{Y} > 0$ such that

$$\begin{aligned} &\bar{Y}(\bar{A} + \bar{B}\hat{G}_1\bar{M}) + (\bar{A} + \bar{B}\hat{G}_1\bar{M})^T\bar{Y} \\ &\qquad + (\bar{C} + \bar{H}\hat{G}_1\bar{M})^T(\bar{C} + \bar{H}\hat{G}_1\bar{M}) < 0 \end{aligned} \tag{14}$$

and

$$(\bar{D} + \bar{E}\hat{G}_2)^T\bar{Y}(\bar{D} + \bar{E}\hat{G}_2) < \gamma^2 I \tag{15}$$

where

$$\bar{Y} = \begin{bmatrix} Y & Y_{12} \\ Y_{12}^T & Y_{22} \end{bmatrix} \in \mathbb{R}^{(n+\widehat{n})\times(n+\widehat{n})}.$$

Using the Schur complement formula, the matrix inequality (14) is equivalent to

$$\Gamma\hat{G}_1\Lambda + (\Gamma\hat{G}_1\Lambda)^T + \Theta < 0 \tag{16}$$

where

$$\Gamma = \begin{bmatrix} \bar{Y}\bar{B} \\ \bar{H} \end{bmatrix},\ \Theta = \begin{bmatrix} \bar{Y}\bar{A} + \bar{A}^T\bar{Y} & \bar{C} \\ \bar{C} & -I \end{bmatrix},\ \Lambda^T = \begin{bmatrix} \bar{M}^T \\ 0 \end{bmatrix}$$

The necessary and sufficient conditions for the LMI (16) to have a solution $\hat{G}_1$ are [8], [14]

$$\begin{aligned} \Gamma^\perp\Theta\Gamma^{\perp T} &< 0 \\ \Lambda^{T\perp}\Theta\Lambda^{T\perp T} &< 0\,. \end{aligned}$$

Noting that $\Gamma^\perp$ and $\Lambda^{T\perp}$ can be selected as follows

$$\Gamma^\perp = \begin{bmatrix} I & 0 & 0 \end{bmatrix} \begin{bmatrix} \bar{Y}^{-1} & 0 \\ 0 & I \end{bmatrix},\ \Lambda^{T\perp} = \begin{bmatrix} I & 0 & 0 \\ 0 & 0 & I \end{bmatrix}$$

these conditions provide (9) and (8) where X is the inverse of the Schur complement of $\bar{Y}$, that is

$$X = (Y - Y_{12}Y_{22}^{-1}Y_{12}^T)^{-1}.$$

Similarly, the matrix inequality (15) has a solution if and only if

$$\bar{E}^\perp(\bar{Y}^{-1} - \frac{1}{\gamma^2}\bar{D}\bar{D}^T)\bar{E}^{\perp T} > 0$$

where $\bar{E}^\perp$ can be selected as

$$\bar{E}^\perp = \begin{bmatrix} I & 0 \end{bmatrix}.$$

Simple matrix algebra provides the condition (7) where X is defined as above. From the definition of X we get

$$Y - X^{-1} = Y_{12}Y_{22}^{-1}Y_{12}^T > 0$$

which is equivalent to (10). Also, the rank in the left-hand-side of (11) is equal to

$$\text{rank} \begin{bmatrix} I & 0 \\ -X^{-1} & I \end{bmatrix} \begin{bmatrix} X & I \\ I & Y \end{bmatrix} \begin{bmatrix} I & -X^{-1} \\ 0 & I \end{bmatrix}$$

$$= \text{rank} \begin{bmatrix} X & 0 \\ 0 & Y - X^{-1} \end{bmatrix} = \text{rank}\ (X) + \text{rank}\ (Y - X^{-1})$$

which provides (11), since

$$\text{rank } (Y - X^{-1}) = \text{rank } (Y_{12}Y_{22}^{-1}Y_{12}^T) = \hat{n}$$

where $Y_{22} \in \mathbb{R}^{\widehat{n}\times\widehat{n}}$. □

Given a feasible solution pair (X, Y) an explicit parametrization of all reduced-order models that correspond this solution can be obtained as follows:

Theorem 2 *All γ-suboptimal $\widehat{n}$th-order models that correspond to a feasible matrix pair (X, Y) are parametrized as*

$$\begin{bmatrix} \hat{C} \\ \hat{A} \end{bmatrix} = -R^{-1}\Gamma^T\Phi\Lambda^T(\Lambda\Phi\Lambda^T)^{-1} + S^{1/2}L(\Lambda\Phi\Lambda^T)^{-1/2} \quad (17)$$

$$\widehat{B} = -Y_{22}^{-1}Y_{12}^T B + Y_{22}^{-1/2}N(\gamma^2 I - B^T X^{-1}B)^{1/2} \quad (18)$$

where

$$S = R^{-1} - R^{-1}\Gamma^T[\Phi - \Phi\Lambda^T(\Lambda\Phi\Lambda^T)^{-1}\Lambda\Phi]\Gamma R^{-1} \quad (19)$$

$$\Phi = (\Gamma R^{-1}\Gamma^T - \Theta)^{-1} \quad (20)$$

$$\Gamma = \begin{bmatrix} 0 & Y_{12} \\ 0 & Y_{22} \\ -I & 0 \end{bmatrix}, \; \Lambda^T = \begin{bmatrix} 0 \\ I \\ 0 \end{bmatrix},$$
$$\Theta = \begin{bmatrix} YA + A^TY & A^TY_{12} & C^T \\ Y_{12}^T A & 0 & 0 \\ C & 0 & -I \end{bmatrix} \quad (21)$$

where L, N are matrices such that $\|L\| < 1$ and $\|N\| < 1$, and R is any positive definite matrix such that $\Phi > 0$, and Y_{12}, Y_{22} are arbitrary matrices such that

$$Y_{12}Y_{22}^{-1}Y_{12}^T = Y - X^{-1} > 0$$

Proof The reduced-order models that correspond to a feasible matrix pair (X, Y) of the γ-suboptimal L_2 model reduction conditions (7)-(11) are obtained from the parametrization of all solutions of the LMIs (16) and (15). Following the results in [8], [14], the set of all solutions of the LMI (16) is given by (17)-(18) where S and Φ as in (19) and (20). Similarly, the set of all solutions of (15) is parametrized by

$$\hat{B} = -\bar{D}^T\bar{Y}\bar{E}(\bar{E}^T\bar{Y}\bar{E})^{-1} + \Psi^{1/2}N\Omega^{1/2}$$

where

$$\begin{aligned} \Psi &= \gamma^2 I - \bar{D}^T\bar{Y}\bar{D} + \bar{D}^T\bar{Y}\bar{E}(\bar{E}^T\bar{Y}\bar{E})^{-1}\bar{E}^T\bar{Y}\bar{D} \\ \Omega &= (\bar{E}^T\bar{Y}\bar{E})^{-1} \end{aligned}$$

Utilizing the definitions (12)-(13) we obtain after a series of matrix algebraic manipulations the parametrization (18). □

According to the above results, the γ-suboptimal L_2 model reduction problem can be characterized as a feasibility problem of finding a pair of positive definite matrices (X, Y) in the intersection of the constraint sets (7), (8), (9), (10) and (11). The constraints (7)-(10) are convex LMIs, but the coupling rank constraint set (11) is non-convex. Alternating projection techniques to address this problem has been developed in [3], [4].

The solution of the optimal L_2 model reduction problem is obtained by solving the following non-convex minimization problem

$$\underset{X,\,Y}{\text{minimize}} \; \gamma \quad (22)$$

subject to the constraints (7)-(11).

3. Discrete-Time Case

Next, we consider the *discrete-time L_2 model reduction problem*: Given a stable, *nth*-order discrete-time system, find a stable $\widehat{n}$*th*-order discrete-time system $\widehat{G}$

$$\widehat{x}_{k+1} = \widehat{A}\widehat{x}_k + \widehat{B}u_k \quad (23)$$
$$\widehat{y}_k = \widehat{C}\widehat{x}_k + \widehat{D}u_k \quad (24)$$

such that the L_2 error

$$\|y - \widehat{y}\|_2 = \left\{\sum_{k=0}^{\infty} \|y(k) - \widehat{y}(k)\|^2\right\}^{1/.2} \quad (25)$$

is minimized, for all impulsive inputs $w \in \mathcal{W}$. That is, the disturbance set $\mathcal{W}$ is the set

$$\mathcal{W} = \{w_0\delta(k) \;\; : w_0 \in \mathbb{R}^m, \; \|w_0\| \le 1\}$$

where $\delta(\cdot)$ is the unit pulse sequence . The *discrete-time γ-suboptimal L_2 model reduction* is defined accordingly.

The following results provide necessary and sufficient conditions for the existence of a solution to the discrete-time γ-suboptimal L_2 model reduction problem and a state-space parametrization of all reduced-order models.

Theorem 3 *There exists an $\widehat{n}$th-order system $\widehat{G}$ to solve the discrete-time γ-suboptimal L_2 model reduction problem if and only if there exist matrices $X > 0$ and $Y > 0$ such that the following conditions are satisfied*

$$X > \frac{1}{\gamma^2}BB^T \quad (26)$$

$$X - AXA^T > 0 \quad (27)$$

$$Y - A^TYA - C^TC > 0 \quad (28)$$

$$\begin{bmatrix} X & I \\ I & Y \end{bmatrix} \ge 0 \quad (29)$$

and

$$\textit{rank} \begin{bmatrix} X & I \\ I & Y \end{bmatrix} \le n + \widehat{n} \; . \quad (30)$$

Proof Consider the error system $\tilde{G} = G - \widehat{G}$ with state space representation

$$\begin{aligned} \dot{\tilde{x}} &= \tilde{A}\tilde{x} + \tilde{B}w \\ z &= \tilde{C}\tilde{x} + \tilde{D}w \end{aligned}$$

where

$$\tilde{x} = \begin{bmatrix} x \\ \hat{x} \end{bmatrix}, \; z = y - \hat{y}$$

$$\begin{aligned} \tilde{A} &= \bar{A} + \bar{B}\hat{G}_1\bar{M} \\ \tilde{B} &= \bar{D} + \bar{B}\hat{G}_2 \\ \tilde{C} &= \bar{C} + \bar{H}\hat{G}_1\bar{M} \\ \tilde{D} &= D + \bar{H}\hat{G}_2 \end{aligned}$$

and

$$\hat{G}_1 = \begin{bmatrix} \hat{C} \\ \hat{A} \end{bmatrix}, \; \hat{G}_2 = \begin{bmatrix} \hat{D} \\ \hat{B} \end{bmatrix}$$

$$\bar{A} = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}, \; \bar{B} = \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix}, \; \bar{D} = \begin{bmatrix} B \\ 0 \end{bmatrix}$$

$$\overline{M} = \begin{bmatrix} 0 & I \end{bmatrix}, \; \overline{H} = \begin{bmatrix} -I & 0 \end{bmatrix}, \; \bar{C} = \begin{bmatrix} C & 0 \end{bmatrix}$$

The discrete-time γ-suboptimal L_2 model reduction problem has a solution if and only if there exists a matrix $\bar{Y} > 0$ such that

$$(\bar{A} + \bar{B}\hat{G}_1\bar{M})^T\bar{Y}(\bar{A} + \bar{B}\hat{G}_1\bar{M}) + (\bar{C} + \bar{H}\hat{G}_1\bar{M})^T(\bar{C} + \bar{H}\hat{G}_1\bar{M}) < \bar{Y} \quad (31)$$

$$(\bar{D} + \bar{E}\hat{G}_2)^T\bar{Y}(\bar{D} + \bar{E}\hat{G}_2) + (\bar{D} + \bar{H}\hat{G}_2)^T(\bar{D} + \bar{H}\hat{G}_2) < \gamma^2 I \quad (32)$$

where

$$\bar{Y} = \begin{bmatrix} Y & Y_{12} \\ Y_{12}^T & Y_{22} \end{bmatrix} \in \mathbb{R}^{(n+\hat{n})\times(n+\hat{n})}$$

Using the Schur complement formula, the matrix inequality (31) is equivalent to

$$(\Upsilon + \Delta\hat{G}_1\bar{M})^T\Sigma(\Upsilon + \Delta\hat{G}_1\bar{M}) < \bar{Y} \quad (33)$$

where

$$\Upsilon = \begin{bmatrix} \bar{A} \\ \bar{C} \end{bmatrix}, \; \Delta = \begin{bmatrix} \bar{B} \\ \bar{H} \end{bmatrix}, \; \Sigma = \begin{bmatrix} \bar{Y} & 0 \\ 0 & I \end{bmatrix}$$

The necessary and sufficient conditions for the matrix inequality (33) to have a solution $\hat{G}_1$ are

$$\begin{aligned} \bar{M}^{T\perp}(\tilde{Y} - \Upsilon^T\Sigma\Upsilon)\bar{M}^{T\perp T} &> 0 \\ \Delta^{\perp}(\Upsilon^{-1} - \Upsilon\tilde{Y}^{-1}\Upsilon^T)\Delta^{\perp T} &> 0 \end{aligned}$$

which are equivalent to (27) and (28). The matrix inequality (32) is equivalent to

$$(\Omega + \Delta\hat{G}_2)^T\Sigma(\Omega + \Delta\hat{G}_1) < \gamma^2 I$$

and there exists a solution $\hat{G}_2$ if and only if

$$\Delta^{\perp}(\Upsilon^{-1} - \frac{1}{\gamma^2}\Omega\Omega^T)\Delta^{\perp T} > 0 \quad (34)$$

which provides (26) where X defined as

$$X = (Y - Y_{12}Y_{22}^{-1}Y_{12}^T)^{-1}$$

Conditions (29) and (30) are obtained as in the continuous time case. □

Theorem 4 *All γ-suboptimal $\widehat{n}$th-order models that correspond to a feasible matrix pair (X, Y) are given by*

$$\begin{bmatrix} \hat{C} \\ \hat{A} \end{bmatrix} = -(\bar{M}Q^{-1}\bar{M}^T)^{-1}\bar{M}Q^{-1}N + (\bar{M}Q^{-1}\bar{M}^T)^{-1/2}K\Psi^{1/2} \quad (35)$$

$$\begin{bmatrix} \widehat{D} \\ \widehat{B} \end{bmatrix} = \begin{bmatrix} D \\ -Y_{22}^{-1}Y_{12}^T B \end{bmatrix} + \begin{bmatrix} I & 0 \\ 0 & Y_{22}^{-1/2} \end{bmatrix} L(\gamma^2 I - B^T X^{-1} B)^{1/2} \quad (36)$$

where

$$\Psi = R^{-1} - N^T Q^{-1} N + N^T Q^{-1}\bar{M}(\bar{M}Q^{-1}\bar{M}^T)^{-1}\bar{M}Q^{-1}N$$

$$Q = \begin{bmatrix} Y - A^T X^{-1} A & Y_{12} \\ Y_{12}^T & Y_{22} \end{bmatrix}, \; \bar{M} = \begin{bmatrix} 0 & I \end{bmatrix}$$

$$N = \begin{bmatrix} -C^T & A^T Y_{12} \\ 0 & 0 \end{bmatrix}, \; R = \begin{bmatrix} I & 0 \\ 0 & Y_{22} \end{bmatrix}$$

where $\|K\| < 1$, $\|L\| < 1$ and Y_{12}, Y_{22} are arbitrary matrices such that

$$Y_{12}Y_{22}^{-1}Y_{12}^T = Y - X^{-1} > 0$$

Proof Follows from the necessary and sufficient solvability conditions of (33) and (34).

4. L_2 to L_∞ Model Reduction

Given a stable *nth*-order strictly proper linear time-invariant system G, the *optimal L_2 to L_∞ gain model reduction problem* is to find a stable $\widehat{n}$*th*-order system $\widehat{G}$ with $\widehat{n} < n$, such that the L_2 to L_∞ gain error

$$\|G - \widehat{G}\|_{L_2 - L_\infty} = \sup_{\|w\|_2 \le 1} \|y - \widehat{y}\|_\infty \quad (37)$$

is minimized, where

$$\|y - \widehat{y}\|_\infty = \sup \|y(t) - \widehat{y}(t)\|$$

and

$$\|w\|_2 = \left\{ \int_0^\infty \|w(t)\|^2 dt \right\}^{1/2}$$

That is, the optimal L_2 to L_∞ model reduction minimizes the peak value of the output error when both models are excited by a bounded energy signal. The *γ-suboptimal L_2 to L_∞ model reduction problem* is to find $\widehat{G}$, if exists, such that

$$\|G - \widehat{G}\|_{L_2 - L_\infty} < \gamma \tag{38}$$

where γ is a given positive scalar.

The following results provide necessary and sufficient conditions for the solution of the γ-suboptimal $L_2 - L_\infty$ model reduction problem in terms of LMIs, and an explicit parametrization of all reduced-order models that correspond to a feasible solution. The proofs follow the same lines as in the L_2 model reduction case and they have been omitted for brevity.

Theorem 5 *There exists an $\hat{n}$th-order system $\widehat{G}$ to solve the γ-suboptimal L_2 to L_∞ model reduction problem if and only if there exist matrices $X > 0$ and $Y > 0$ such that the following conditions are satisfied*

$$AX + XA^T + BB^T < 0 \tag{39}$$

$$Y - \frac{1}{\gamma^2}C^TC > 0 \tag{40}$$

$$YA + A^TY < 0 \tag{41}$$

$$\begin{bmatrix} X & I \\ I & Y \end{bmatrix} \geq 0$$

and

$$\text{rank} \begin{bmatrix} X & I \\ I & Y \end{bmatrix} \leq n + \hat{n} \; .$$

Theorem 6 *All γ-suboptimal $\hat{n}$th-order models that correspond to a feasible matrix pair (X, Y) are given by*

$$\begin{bmatrix} \widehat{B} & \widehat{A} \end{bmatrix} = R^{-1}\Gamma^T\Phi\Lambda^T(\Lambda\Phi\Lambda^T)^{-1} + S^{1/2}L(\Lambda\Phi\Lambda^T)^{-1/2}$$

$$\widehat{C} = -CX_{12}^TX_{22}^{-1} + (\gamma^2 I - CX^{-1}C^T)^{1/2}LX_{22}^{-1}$$

where

$$S = R^{-1} - R^{-1}\Gamma^T[\Phi - \Phi\Lambda^T(\Lambda\Phi\Lambda^T)^{-1}\Lambda\Phi]\Gamma R^{-1}$$

$$\Phi = (\Gamma R^{-1}\Gamma^T - \Theta)^{-1}$$

$$\Gamma = \begin{bmatrix} 0 \\ I \\ 0 \end{bmatrix}, \; \Lambda^T = \begin{bmatrix} 0 & X_{12} \\ 0 & X_{22} \\ I & 0 \end{bmatrix},$$

$$\Theta = \begin{bmatrix} AX + XA^T & AX_{12} & B \\ X_{12}^TA^T & 0 & 0 \\ B^T & 0 & -I \end{bmatrix}$$

where $L \in \mathbb{R}^{(p+\hat{n})\times(m+\hat{n})}$ is any matrix such that $\|L\| < 1$, R is any positive definite matrix such that $\Phi > 0$, and X_{12}, X_{22} are arbitrary matrices such that

$$X_{12}X_{22}^{-1}X_{22}^T = X - Y^{-1} > 0$$

The discrete-time L_2 to L_∞ model reduction is defined is a similar way. The following results show the necessary and sufficient conditions for the solution of the discrete-time γ-suboptimal $L_2 - L_\infty$ model reduction problem and the parametrization of all reduced-order models.

Theorem 7 *There exists an $\hat{n}$th-order system $\widehat{G}$ to solve the discrete-time γ-suboptimal L_2 to L_∞ model reduction problem if and only if there exist matrices $X > 0$ and $Y > 0$ such that the following conditions are satisfied*

$$X - AXA^T - BB^T > 0 \tag{42}$$

$$Y - \frac{1}{\gamma^2}C^TC > 0 \tag{43}$$

$$Y - A^TYA > 0$$

$$\begin{bmatrix} X & I \\ I & Y \end{bmatrix} \geq 0 \tag{44}$$

and

$$\text{rank} \begin{bmatrix} X & I \\ I & Y \end{bmatrix} \leq n + \hat{n} \; . \tag{45}$$

Theorem 8 *All γ-suboptimal $\hat{n}$th-order models that correspond to a feasible matrix pair (X, Y) are given by*

$$\begin{bmatrix} \widehat{D} & \widehat{C} \end{bmatrix} = -(\bar{E}^TQ^{-1}\bar{E})^{-1}\bar{E}Q^{-1}N + (\bar{E}^TQ^{-1}\bar{E})^{-1/2}K\Psi^{1/2} \tag{46}$$

$$\begin{bmatrix} \widehat{B} & \widehat{A} \end{bmatrix} = \begin{bmatrix} D & CX_{12}X_{22}^{-1} \end{bmatrix} + (\gamma^2 I - CY^{-1}C^T)^{1/2}L\begin{bmatrix} I & 0 \\ 0 & X_{22}^{-1/2} \end{bmatrix} \tag{47}$$

where

$$\Psi = R^{-1} - N^TQ^{-1}N + N^TQ^{-1}\bar{E}(\bar{E}^TQ^{-1}\bar{E})^{-1}\bar{E}Q^{-1}N$$

$$Q = \begin{bmatrix} X - AY^{-1}A^T & X_{12} \\ X_{12}^T & X_{22} \end{bmatrix}, \; \bar{E} = \begin{bmatrix} 0 \\ I \end{bmatrix}$$

$$N = \begin{bmatrix} B^T & A^TY_{12}Y_{22}^{-1} \\ 0 & 0 \end{bmatrix}, \; R = \begin{bmatrix} I & 0 \\ 0 & X_{22} \end{bmatrix}$$

where $\|K\| < 1$, $\|L\| < 1$ and X_{12}, X_{22} are arbitrary matrices such that

$$X_{12}X_{22}^{-1}X_{22}^T = X - Y^{-1} > 0$$

5. Numerical Example

As an example consider the following system that has been examined in [11], [7] and [18]

$$\begin{aligned} \dot{x} &= \begin{bmatrix} -0.005 & -0.99 \\ -0.99 & -5000 \end{bmatrix} x + \begin{bmatrix} 1 \\ 100 \end{bmatrix} w \\ y &= \begin{bmatrix} 1 & 100 \end{bmatrix} x \end{aligned}$$

The balanced model reduction method produces a reduced-order model

$$\begin{aligned}\dot{\hat{x}} &= -0.005\hat{x} + w \\ \hat{y} &= \hat{x}\end{aligned}$$

with L_2 error equal to $\|y - \hat{y}\|_2 = 100.0$. Utilizing an alternating projection algorithm [3], [4] along with a bisection method to optimize the L_2 error bound γ, the following optimal L_2 reduced-order model was obtained

$$\begin{aligned}\dot{\hat{x}} &= -4999.8\hat{x} - 100.0w \\ \hat{y} &= -100.0\hat{x}\end{aligned}$$

with L_2 error equal to $\|y - \hat{y}\|_2 = 9.8$. The corresponding matrix parameters (X, Y) that satisfy the conditions (7)-(11) are

$$X = \begin{bmatrix} 0.1161 & 0.0312 \\ 0.0312 & 108.9409 \end{bmatrix}, \; Y = \begin{bmatrix} 8.6139 & 0.0003 \\ 0.0003 & 1.3048 \end{bmatrix}$$

Figure 1 shows the reduction of the error of the alternating projection algorithm versus the iteration number.

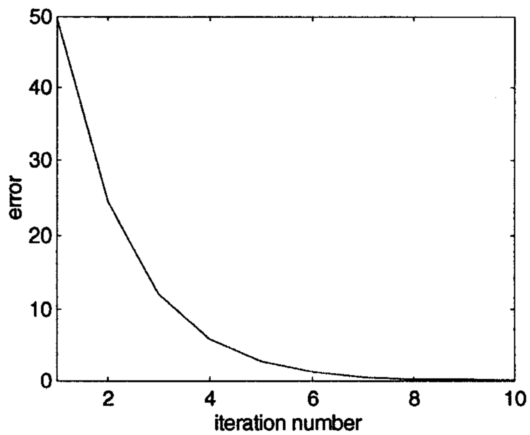

Fig. 1: Error of Alternating Project Algorithm vs Iteration number

It is noted that, in general, convergence of the alternating projection approach to the global optimum solution is not guaranteed.

6. Conclusion

An explicit characterization of the solution to the L_2 and $L_2 - L_\infty$ model reduction problems was provided. Necessary and sufficient conditions in terms of LMIs and a non-convex coupling constraint were obtained, and a parametrization of all reduced-order models that correspond to a feasible matrix solution was derived. Both continuous-time and discrete-time results were presented. The solution of a non-convex feasibility problem is required to compute reduced-order models and alternating projection algorithms are proposed to address the computational issue, although global convergence of the technique is not guaranteed.

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