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Robust Control with Variance Finite-signal-to-noise Models

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Abstract

The newly introduced Finite Signal-to-Noise (FSN) models can model two kinds of uncertainties in controlled linear systems: the so-called finite-signal-to-noise ratio and multiplicative noises. FSN models uncertainties as white noises of intensity depending affinely on some signal variances in the controlled systems. This paper provides a complete solution for the analysis and synthesis of the linear system with FSN model uncertainty. The linear variance operator is a key mechanism used to deduce results for stability and performance robust to these uncertainties.

1 Introduction

The robust H_2 control studies the control problem with respect to norm-bounded uncertainties. This problem is the ultimate goal of many robust control researches. Although [1] provides a rigorous solution to robust H_2 performance analysis problem, the synthesis problem is open. A more practical and also much harder problem is the robust H_2 control problem with respect to parametric uncertainties. Very little is known for this problem [2]. As an alternative approach, the effect of parametric uncertainty is studied by using stochastic parametric uncertainty description [3, 4], which is also termed by *multiplicative noise* [5]. This paper studies the robust H_2 control with respect to the newly introduced representation of finite-signal-to-noise uncertainty [7]. FSN models uncertainties as white noises of intensity depending affinely on the variances or covariances of some signals in the controlled systems. The former is called *variance* FSN or VFSN model, and the latter is called *covariance* FSN or CFSN model. The full-state feedback control synthesis for CFSN uncertainty is studied in [8, 9], and [6] mainly studied the control problem for VFSN uncertainty. It is not hard to check that the multiplicative noise is a special case of the CFSN uncertainty, while VFSN models studied in [6] might not be a reasonable description for multiplicative noises. However, as shown in [6], the control problem for VFSN uncertainty is computationally more tractable than for CFSN uncertainty.

In this paper, we studied a special VFSN uncertainty structure which can be used to model multiplicative noise. This VFSN structure is a stochastic counterpart of deterministic models with diagonal parameter perturbations, and it also models the FSN noise in a channel by channel case. A key concept of a *variance operator* is used to deduce results of stability and performance with respect to VFSN uncertainties. The stability considered here is simply the finiteness of the variance in each channel, which is proven to be equivalent to *mean square stability* studied in [4, 6].

Although the analysis of VFSN uncertainty can be conducted by the LMI approach studied in [4], the analysis conducted here aims for output feedback control synthesis. The computation of the LMI approach in [4] is much involved than the eigenvalue computation used in this paper. The optimal controller, which optimizes the robustness measure μ_{fsn} considered in this paper, is proven to be one that solves an optimal variance control problem with properly chosen weight and disturbance intensity. Notice that although we carried out our study on discrete time system in this paper, there is no difficulty to extend it to continuous time systems.

The paper is organized as the follows. Section 2 describes the FSN model. In section 3, both VFSN stability and VFSN performance analysis are studied by introducing the variance operator. Section 4 presents the output feedback control synthesis, the resulting controller is robust to VFSN uncertainty. An algorithm called *Q− W iteration* is introduced. Finally in section 5, a numerical example is studied.

The following notations are used in this paper. R, R_+ $\triangleq$ sets of real and positive real numbers; $\mathsf{P}^{n\times n}$ $\triangleq$ set of $n \times n$ positive definite matrices; $[\cdot]_{ij}$ $\triangleq$ (i, j)-element of matrix $[\cdot]$; bdiag $\triangleq$ block diagonal operation of matrices ; E_i $\triangleq$ matrix with 1 at its (i, i)-location and 0

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at all other locations; $\bar{\lambda}(\cdot) \triangleq$ the largest eigenvalue of a matrix $(\cdot)$; $\mathcal{E}_\infty \triangleq$ steady state expectation operator.

2 Finite-signal-to-noise Model

Consider the following MIMO linear discrete time system

$$\begin{bmatrix} \tilde{z} \\ \partial x \end{bmatrix} = \begin{bmatrix} \tilde{D} & C \\ B & A \end{bmatrix} \begin{bmatrix} \tilde{w} \\ x \end{bmatrix} \tag{1}$$

where ∂ is the delay operator, $\tilde{z} \in \mathsf{R}^{p+n}$ and $\tilde{w} \in \mathsf{R}^{q+n}$. We partition $\tilde{w}$ and $\tilde{z}$ as

$$\tilde{z} = \begin{bmatrix} z_0^T & z^T \end{bmatrix}^T, \quad \tilde{w} = \begin{bmatrix} w_0^T & w^T \end{bmatrix}^T$$

and partition z and w as

$$z = \begin{bmatrix} z_1 & z_2 & \cdots & z_n \end{bmatrix}^T, \quad w = \begin{bmatrix} w_1 & w_2 & \cdots & w_n \end{bmatrix}^T$$

such that $z_0 \in \mathsf{R}^p$, $w_0 \in \mathsf{R}^q$ and $w_i, z_i \in \mathsf{R}$ for $i = 1, 2, \cdots, n$. We also partition C and $\tilde{D}$ according to the dimensions of z_i for $i = 0, 1, \cdots, n$

$$C = \begin{bmatrix} C_0^T & C_1^T & \cdots & C_n^T \end{bmatrix}^T$$
$$\tilde{D} = \begin{bmatrix} \tilde{D}_0^T & \tilde{D}_1^T & \cdots & \tilde{D}_n^T \end{bmatrix}^T$$

and partition B and $\tilde{D}_j$ according to the dimensions of w_i for $j = 0, 1, \cdots, n$

$$\begin{aligned} B &= \begin{bmatrix} B_0 & B_1 & \cdots & B_n \end{bmatrix} \\ \tilde{D}_j &= \begin{bmatrix} D_{j0} & D_{j1} & \cdots & D_{jn} \end{bmatrix}. \end{aligned}$$

Definition 2.1: *The linear system (1) is said to have variance finite-signal-to-noise (VFSN) uncertainty structure if the following are true*

(i) *w_i are independent zero-mean stochastic processes for $i = 0, 1, \cdots, n$.*

(ii) *the following holds for $i = 1, 2, \ldots, n$*

$$\mathcal{E}_\infty w_i^2 = \sigma_i \mathcal{E}_\infty z_i^2 \tag{2}$$

where $\sigma_i^{-\frac{1}{2}}$'s for $i = 1, 2, \ldots, n$ are the VFSN ratios.

denote the VFSN ratio vector by the element by element inverse of the square root of $\sigma = \begin{bmatrix} \sigma_1 & \sigma_2 & \cdots & \sigma_n \end{bmatrix}^T$ and assume that σ is unknown but lying within a convex set (convex polytope) Σ in R_+^n.

In this paper, we only consider the case where the w_i's are mutually independent white noise processes. Without loss of generality, we consider σ_i^+'s as known upper bounds, and

$$\Sigma = \{\sigma \ : \ 0 \le \sigma_i \le \sigma_i^+, \ i = 1, 2, \cdots, n\}.$$

3 Robustness Analysis

Consider the system (1). Assume it is asymptotically stable. Denote the observability gramian of (1) associated with only the ith output as L_i which solves

$$L_i = A^T L_i A + C_i^T C_i. \tag{3}$$

Since (1) is stable, hence (3) has a unique and positive definite solution for each i.

Definition 3.1: *Let $\tilde{W} \in \mathsf{P}^{(q+n)\times(q+n)}$ be the covariance of the input signal $\tilde{w}$. The variance of the output z_i can be thought of as an operator from $\mathsf{P}^{(q+n)\times(q+n)}$ to R_+, denoted by*

$$\mathrm{Var}_i(\tilde{W}) = \mathcal{E}_\infty z_i^2, \quad i = 1, 2, \cdots, n.$$

By stacking these Var_i's into a vector, we obtain a vector-valued operator $\mathrm{Var} \ : \ \mathsf{P}^{(q+n)\times(q+n)} \to \mathsf{R}_+^n$ of the following form

$$\mathrm{Var}(\tilde{W}) = \begin{bmatrix} \mathrm{Var}_1(\tilde{W}) \\ \mathrm{Var}_2(\tilde{W}) \\ \vdots \\ \mathrm{Var}_n(\tilde{W}) \end{bmatrix}, \quad \tilde{W} \in \mathsf{P}^{(q+n)\times(q+n)}$$

we call it the variance operator. $\mathrm{Var}_i(\tilde{W})$ can be computed from the observability gramian satisfying (3)

$$\mathrm{Var}_i(\tilde{W}) = \mathrm{tr}[\tilde{W}(\tilde{D}_i^T \tilde{D}_i + B^T L_i B)]$$

hence the variance operator Var is a linear mapping from $\mathsf{P}^{(q+n)\times(q+n)} \to \mathsf{R}_+^n$.

Now we consider the case where $w_i (i = 1, 2, \cdots, n)$ have VFSN structures. If we denote the ith element Z_i of a vector $Z \in \mathsf{R}_+^n$ as the variance of the ith output of z_i for $i = 1, 2, \cdots, n$, and the covariances of w as W, then the VFSN structure implies

$$W = \Delta \, \mathrm{diag}(Z)$$

where $\Delta \in \mathbf{\Delta}$ and

$$\mathbf{\Delta} = \{\mathrm{diag}(\sigma) \ : \ \sigma \in \Sigma\}. \tag{4}$$

In this case $Z = \begin{bmatrix} Z_1 & Z_2 & \cdots & Z_n \end{bmatrix}^T \in \mathsf{R}_+^n$ will satisfy

$$Z = \mathrm{Var}\left(\begin{bmatrix} W_0 & 0 \\ 0 & \Delta \, \mathrm{diag}(Z) \end{bmatrix}\right) = \mathcal{V}(Z) \tag{5}$$

where $\mathcal{V}(\cdot)$ is a linear mapping from R_+^n to R_+^n. Using this notation, (5) simply implies that Z is a fixed point of $\mathcal{V}(\cdot)$ in R_+^n. Notice that $\mathcal{V}(\cdot)$ is also a function of Δ. However for notational simplicity, we do not explicitly address this dependence which could be determined from the context. Also notice that the fixed point Z of $\mathcal{V}(\cdot)$ satisfying (5) depends on $\Delta \in \mathbf{\Delta}$.

Definition 3.2: *The system (1) is said to be VFSN stable with respect to* $\mathbf{\Delta}$*, if for any* $\Delta \in \mathbf{\Delta}$*, there exists a unique and finite vector* $Z \in \mathrm{R}_+^n$ *satisfying (5).*

Denote the transfer function from w to z of system (1) as $T(s)$. The following theorem provides conditions to address the VFSN stability.

Theorem 3.3: *The following statements are equivalent*

(i) *(1) is VFSN stable with respect to* $\mathbf{\Delta}$.

(ii) $\forall\ \Delta \in \mathbf{\Delta}$, $\mathcal{V}(\cdot)\ :\ \mathrm{R}_+^n \to \mathrm{R}_+^n$ *is a contraction in* R_+^n*, i.e.,* $\forall\ Z, \hat{Z} \in \mathrm{R}_+^n$,

$$\|\mathcal{V}(Z) - \mathcal{V}(\hat{Z})\| < \|\tilde{Z}\|$$

for some vector norm $\|\cdot\|$ *equipped by* R^n.

(iii) $\mu_{\text{fsn}}(T, \mathbf{\Delta}) < 1$*, where the measure*

$$\mu_{\text{fsn}}(T, \mathbf{\Delta}) = \max_{\Delta \in \mathbf{\Delta}} \bar{\lambda}(\mathtt{G}\Delta)$$

and

$$\mathtt{G} = \{\mathtt{G}_{ij}\},\ \ \mathtt{G}_{ij} = \mathrm{Var}_i(E_{q+j}).$$

(iv) *For any given* $\Delta \in \mathbf{\Delta}$*, there exists a* $X = X^T > 0$ *and*

$$Z = \begin{bmatrix} Z_1 & Z_2 & \cdots & Z_n \end{bmatrix}^T \in \mathrm{R}_+^n$$

satisfying

$$X = AXA^T + B_0W_0B_0^T + \sum_{i=1}^n \sigma_i Z_i B_i B_i^T \tag{6}$$

$$Z_j = C_jXC_j^T + D_{j0}W_0D_{j0}^T + \sum_{i=1}^n \sigma_i Z_i D_{ji}^2 \tag{7}$$

for $j = 1, 2, \cdots, n$.

If the above conditions hold, then the measure $\mu_{\text{fsn}}(T, \mathbf{\Delta})$ *can be computed as*

$$\mu_{\text{fsn}}(T, \mathbf{\Delta}) = \bar{\lambda}(\mathtt{G}\Delta^+) \tag{8}$$

where $\Delta^+ = \mathrm{diag}(\sigma_1^+, \sigma_2^+, \cdots, \sigma_n^+)$.

Proof: $\forall\ Z, \hat{Z} \in \mathrm{R}_+^n$, define $\tilde{Z} = Z - \hat{Z}$. Consider the difference

$$\begin{aligned} \mathcal{V}(Z) - \mathcal{V}(\hat{Z}) &= \mathrm{Var}(\begin{bmatrix} 0 & 0 \\ 0 & \Delta\,\mathrm{diag}(\tilde{Z}) \end{bmatrix}) \\ &= \mathrm{Var}(\sum_{i=1}^n \delta_i \tilde{Z}_i E_{q+i}) \\ &= \sum_{i=1}^n \delta_i \tilde{Z}_i \mathrm{Var}(E_{q+i}) \\ &= \mathtt{G}\Delta\tilde{Z}. \end{aligned}$$

(ii) implies that for all $\Delta \in \mathbf{\Delta}$

$$\|\mathtt{G}\Delta\tilde{Z}\| < \|\tilde{Z}\|,\ \ \forall \tilde{Z} \neq 0$$

or for all $\Delta \in \mathbf{\Delta}$ there exists an induced matrix norm $\|\cdot\|_{in}$ associated with the vector norm $\|\cdot\|$ such that $\|\mathtt{G}\Delta\|_{in} < 1$. Since the matrix spectral radius is less than any induced matrix norm, this is equivalent to $\max_{\Delta \in \mathbf{\Delta}} \rho(\mathtt{G}\Delta) < 1$. Consider that $\mathtt{G}\Delta$ is a matrix of all positive elements, by the Perron-Frobenius theorem [12], $\rho(\mathtt{G}\Delta) = \bar{\lambda}(\mathtt{G}\Delta)$. Hence (ii) is equivalent to (iii). (ii) $\Rightarrow$ (i) is a standard result, hence (iii) $\Rightarrow$ (i). Now we assume that (iii) fails, i.e., there exists a $\Delta_0 \in \mathbf{\Delta}$ such that $\bar{\lambda}(\mathtt{G}\Delta_0) \not< 1$. Since $0 \in \mathbf{\Delta}$ and $\bar{\lambda}(\mathtt{G}\Delta)$ is a continuous function of Δ, hence there must exist another $\Delta_1 \in \mathbf{\Delta}$ such that $\bar{\lambda}(\mathtt{G}\Delta_1) = 1$, or $\det(I - \mathtt{G}\Delta_1) = 0$. This implies that the following equations will not have a unique solution for Z

$$Z = \mathrm{Var}(\begin{bmatrix} W_0 & 0 \\ 0 & \Delta_1\,\mathrm{diag}(Z) \end{bmatrix}).$$

Hence if (iii) fails, then $\mathcal{V}(\cdot)$ does not have a unique solution for Z, i.e., (i) fails. Up to now, we proved that (ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (i).

Let (iv) be true. For a given $\Delta \in \mathbf{\Delta}$ the conditions listed in (iv) are linear equations with respect to the unknowns X and Z, and A is asymptotically stable. Hence if there exist a solution for those equations, it must be unique. This implies (iv) $\Rightarrow$ (i). Since A is an asymptotically stable matrix, there must exist $X_i = X_i^T \geq 0$ for $i = 0, 1, \cdots, n$ such that

$$\begin{aligned} X_0 &= AX_0A^T + B_0W_0B_0^T \\ X_i &= AX_iA^T + B_iB_i^T,\ \ i = 1, 2, \cdots, n \end{aligned}$$

If (i) holds, i.e., there exists a finite vector $Z \in \mathrm{R}_+^n$ satisfying

$$Z_i = \mathrm{tr}[W_0(D_{i0}^TD_{i0} + B_0^TL_iB_0)] + \sum_{j=1}^n \mathrm{tr}[W_j(D_{ij}^2 + B_j^TL_iB_j)].$$

Construct a matrix of the following form

$$X = X_0 + \sum_{i=1}^n \sigma_i Z_i X_i \geq 0.$$

It is obvious that this X is finite and satisfies (6) and Z_i satisfies (7). Hence (i) implies (iv).

(8) can be obtained by using the properties of matrix with all elements in R_+^n. □

Remark: $\mu_{\text{fsn}}(T, \mathbf{\Delta})$ provides a quantitative measure to check if (1) has VFSN stability, and reflects the admissible size of VFSN uncertainty for the VFSN stability to remain. Let the stability radius of (1) with respect to VFSN uncertainty be defined as

$$\kappa(T, \mathbf{\Delta}) = \sup\{r\ :\ (1)\text{ is VFSN stable }\forall \tfrac{1}{r}\Delta \in \mathbf{\Delta}\}$$

then we have

$$\kappa(T, \mathbf{\Delta}) = (\mu_{\text{fsn}}(T, \mathbf{\Delta}))^{-1}. \tag{9}$$

Let $\tilde{T}(s)$ be the transfer function matrix from $\tilde{w}$ to $\tilde{z}$ in system (1). Now we consider the VFSN performance of $\tilde{T}(s)$, i.e., the output variance $\mathcal{E}_\infty z_0^T z_0$ of z_0 under w_0 and the VFSN structure (2).

The variance operator $\text{Var}_0(\cdot)$ which maps

$$W_0 \triangleq \mathcal{E}_\infty w_0 w_0^T$$

to R_+ under VFSN structure can be expressed as

$$\mathcal{E}_\infty z_0^T z_0 = \text{Var}_0(\begin{bmatrix} W_0 & 0 \\ 0 & \Delta\,\text{diag}(Z) \end{bmatrix}) = \mathcal{V}_0(Z)$$

which can be further expressed as

$$\mathcal{V}_0(Z) = \text{tr}\{\begin{bmatrix} W_0 & 0 \\ 0 & \Delta\,\text{diag}(Z) \end{bmatrix}(\tilde{D}_0^T\tilde{D}_0 + B^T L_0 B)\} \tag{10}$$

with L_0 satisfying (3) and Z satisfying (5).

Definition 3.4: *(1) has VFSN performance if it is VFSN stable and for all $\Delta \in \mathbf{\Delta}$ the output variance of z_0 in (10) satisfies*

$$\mathcal{V}_0(Z) < \gamma \tag{11}$$

where Z satisfies (5). The worst-case output variance of z_0 over $\mathbf{\Delta}$ is computed as

$$\nu(\tilde{T}, \mathbf{\Delta}) = \max_{\Delta\in\mathbf{\Delta}}\{\mathcal{V}_0(Z)\ :\ Z \text{ satisfies (5) and } \mu(T, \mathbf{\Delta}) < 1\}.$$

Compute

$$\tilde{\mathsf{G}} = \begin{bmatrix} \tilde{\mathsf{G}}_{11} & \tilde{\mathsf{G}}_{12} \\ \tilde{\mathsf{G}}_{21} & \tilde{\mathsf{G}}_{22} \end{bmatrix} \tag{12}$$

$$\tilde{\mathsf{G}}_{11} = \text{Var}_0(\text{bdiag}(W_0, 0))$$
$$\tilde{\mathsf{G}}_{12} = [\ \text{Var}_0(E_{q+1})\ \ \cdots\ \ \text{Var}_0(E_{q+n})\ ]$$
$$\tilde{\mathsf{G}}_{21} = \text{Var}(\text{bdiag}(W_0, 0))$$
$$\tilde{\mathsf{G}}_{22} = \{\mathsf{G}_{ij}\},\ \ \mathsf{G}_{ij} = \text{Var}_i(E_{q+j})$$

then $\mathcal{V}_0(Z)$ can be expressed as the following linear fractional form

$$\mathcal{V}_0(Z) = \tilde{\mathsf{G}}_{11} + \tilde{\mathsf{G}}_{12}\Delta(I - \tilde{\mathsf{G}}_{22}\Delta)^{-1}\tilde{\mathsf{G}}_{21}.$$

Consider the following set

$$\mathbf{\Delta}_\gamma = \{\text{diag}(\sigma_0, \Delta) \mid 0 \le \sigma_0 \le \frac{1}{\gamma},\ \Delta \in \mathbf{\Delta}\}. \tag{13}$$

Theorem 3.5: *The following statements are equivalent:*

(i) *(1) has VFSN performance.*

(ii) $\mu_{\text{fsn}}(\tilde{T}, \mathbf{\Delta}_\gamma) < 1$.

If the above hold, then the worst-case VFSN performance can be computed as

$$Z_0^{\text{worst}} = \tilde{\mathsf{G}}_{11} + \tilde{\mathsf{G}}_{12}\Delta^+(I - \mathsf{G}_{22}\Delta^+)^{-1}\tilde{\mathsf{G}}_{21}$$

where Δ^+ is defined as

$$\Delta^+ = \text{diag}(\sigma_1^+, \cdots, \sigma_n^+). \tag{14}$$

Proof: FSN performance implies that (1) is FSN stable or $\forall\ \Delta \in \mathbf{\Delta}$

$$\det(I - \tilde{\mathsf{G}}_{22}\Delta) > 0$$

and the output variance Z_0 satisfies

$$1 - Z_0\frac{1}{\gamma} > 0$$

which is equivalent to for all $0 \le \delta_0 \le \frac{1}{\gamma}$, $\Delta \in \mathbf{\Delta}$

$$\det(I - \tilde{\mathsf{G}}_{22}\Delta)[1 - \delta_0(\tilde{\mathsf{G}}_{11} + \tilde{\mathsf{G}}_{12}\Delta(I - \tilde{\mathsf{G}}_{22}\Delta)^{-1}\tilde{\mathsf{G}}_{21})] > 0$$

this is equivalent to for all $\tilde{\Delta} \in \mathbf{\Delta}_\gamma$

$$\det(I - \tilde{\mathsf{G}}\tilde{\Delta}) > 0$$

Hence the claim follows.

If (ii) holds, then

$$Z_0 = \tilde{\mathsf{G}}_{11} + \tilde{\mathsf{G}}_{12}[\Delta + \Delta\tilde{\mathsf{G}}\Delta + \Delta\tilde{\mathsf{G}}^2\Delta^2 + \cdots]\tilde{\mathsf{G}}_{21}$$

which implies the worst-case Z_0 occurs at $\Delta = \Delta^+$. □

4 Robust Control Synthesis

Consider the following MIMO discrete time system $P(s)$

$$\begin{bmatrix} z_0 \\ z \\ y \\ \partial x \end{bmatrix} = \begin{bmatrix} D_{00} & D_{01} & D_{02} & C_0 \\ D_{10} & D_{11} & D_{12} & C_1 \\ D_{20} & D_{21} & D_{22} & C_2 \\ B_0 & B_1 & B_2 & A \end{bmatrix}\begin{bmatrix} w_0 \\ w \\ u \\ x \end{bmatrix} \tag{15}$$

which has VFSN structure of the form in (2). In the sequential discussion, we assume (A, B_2, C_2) is a stabilizable and detectable triple. A controller $K(s)$ sought for $P(s)$ is of the following form

$$\begin{bmatrix} u \\ \partial x_c \end{bmatrix} = \begin{bmatrix} D_c & C_c \\ B_c & A_c \end{bmatrix}\begin{bmatrix} y \\ x_c \end{bmatrix}. \tag{16}$$

Let

$$\tilde{w} = [\ w_0^T\ \ w^T\ ]^T,\ \tilde{z} = [\ z_0^T\ \ z^T\ ]^T$$

Denote the closed loop transfer functions from $\tilde{w}$ to $\tilde{z}$, from w to z as $\tilde{T}(P, K)$ and $T(P, K)$ respectively.

Robust Stabilization Problem: *Given $\rho > 0$, find a controller K such that the closed loop system has stability margin at least κ, i.e., the controller K solves the following feasibility problem*

$$\kappa(T(P,K),\mathbf{\Delta}) \geq \rho$$

and the optimal robust stabilizer $K_{\rm opt}(s)$ solves

$$K_{\rm opt}(s) = \arg\max_K \kappa(T(P,K),\mathbf{\Delta}).$$

Robust Performance Problem: *Given a performance level $\gamma > 0$, find a controller $K(s)$ such that the output variance of the closed loop system is bounded by γ for all $\Delta \in \mathbf{\Delta}$, i.e., K solves the following feasibility problem*

$$\mu_{\rm fsn}(\tilde{T}(P,K),\mathbf{\Delta}_\gamma) < 1$$

where $\mathbf{\Delta}_\gamma$ is defined in (13). The optimal robust controller solves

$$K_{\rm opt}(s) = \arg\min_K \mu_{\rm fsn}(\tilde{T}(P,K),\mathbf{\Delta}_\gamma). \quad (17)$$

Due to relationship (9), the robust stabilization problem is a special case of robust performance problem. In the following, we focus on the robust performance problem.

Theorem 4.1: *If a controller $K(s)$ solves the robust performance problem, i.e., $K(s)$ solves (17), then there exists a weight $Q = Q^T \geq 0$ and a white noise disturbance with intensity $W \geq 0$, such that $K(s)$ solves the following weighted variance control problem for system (1)*

$$K(s) = \arg\min_K \mathrm{tr}\,\mathcal{E}_\infty[\tilde{z}^T Q \tilde{z}].$$

Proof: Assume $K(s)$ solves (17), then there exists β such that

$$0 \leq \beta = \min_K \mu_{\rm fsn}(\tilde{T}(P,K),\mathbf{\Delta}_\gamma).$$

Denote $\tilde{\mathtt{G}}$ (see (12) as the matrix computed from the closed loop system, then

$$\beta = \bar{\lambda}(\tilde{\mathtt{G}}\,\mathrm{diag}(\frac{1}{\gamma},\sigma_1^+,\cdots,\sigma_n^+)).$$

Let l and r be the left and right eigenvectors of

$$\tilde{\mathtt{G}}\,\mathrm{diag}(\frac{1}{\gamma},\sigma_1^+,\cdots,\sigma_n^+)$$

associated with β such that l and r are normalized to satisfy

$$l^T r = 1. \quad (18)$$

Since each element of $\mathtt{G}$ belongs to R_+, hence $l, r \in \mathsf{R}_+^{n+1}$ (see [12]). This further implies that there exists $Q(l)$ and $W(r)$ satisfying

$$Q(l) = \mathrm{bdiag}(l_1 I_q, l_2, l_3, \cdots, l_{n+1}) \quad (19)$$
$$W(r) = \mathrm{bdiag}(W_0\frac{r_1}{\gamma}, \sigma_1^+ r_2, \cdots, \sigma_n^+ r_{n+1}) \quad (20)$$

such that

$$\beta = \mathcal{E}_\infty[\tilde{z}^T Q(l)\tilde{z}]$$

where

$$\tilde{z} = [\ z_0^T \quad z^T\ ]^T$$

is the output of the closed loop system $\tilde{T}(P,K)$ with respect to the white noise signal

$$\tilde{w} = [\ w_0^T \quad w^T\ ]^T$$

of the intensity $W(r)$. □

Theorem 4.1 implies that the robust control problem with respect to VFSN uncertainty can be solved via an optimal variance control problem by choosing a proper weight and a white noise disturbance with proper intensity. Hence the computation of the weight Q and intensity W is crucial to solving VFSN control problem. In the following, an iterative algorithm is proposed to compute the possible Q and W, we shall call it *Q-W* iteration.

The *Q-W* Iteration:

(i) *Choose $l_0, r_0 \in \mathsf{R}_+^{n+1}$ satisfying (18) and compute the weight $Q(l_0)$ and intensity $W(r_0)$ as shown in (19) and (20).*

(ii) *Solve the optimal variance control problem (say by using the theorem in appendix B) for controller K of form (16)*

$$\begin{array}{ll} \text{minimize} & \mathcal{E}_\infty[\tilde{z}^T Q(l_0)\tilde{z}] \\ \text{subject to} & (15),\ (16),\ \mathcal{E}_\infty[\tilde{w}\tilde{w}^T] = W(r_0). \end{array}$$

(iii) *Use the optimal variance controller obtained in (ii) to compute the matrix $\tilde{\mathtt{G}}$ (see (12)) of the closed loop system. Also compute the largest eigenvalue β of the matrix*

$$\tilde{\mathtt{G}}\,\mathrm{diag}(\frac{1}{\gamma},\sigma_1^+,\cdots,\sigma_n^+)$$

and the associated left and right eigenvectors $l, r \in \mathsf{R}_+^{n+1}$, which are normalized to satisfy (18). Compute the corresponding $Q(l)$ and $W(r)$ from (19) and (20).

(iv) *If $\|Q(l_0) - Q(l)\| + \|W(r_0) - W(r)\| < \epsilon$, go to step (v). Otherwise, set*

$$\lambda l_0 + (1-\lambda)l \to l_0,\ \lambda r_0 + (1-\lambda)r \to r_0$$

and go to step (i), where ϵ is a given error tolerance and $\lambda \in [0,\ 1]$.

(v) *Set $\mu_{\rm fsn} = \beta$. If $\beta < 1$, then formulate the final controller and stop. If $\beta \geq 1$, γ bound may be too tight, hence stop.*

We say Q-W iteration is successful, if the Q-W iteration converges.

Theorem 4.2: *For a given performance bound γ, if Q-W iteration is successful, then there exists a unique β such that $\mu_{\text{fsn}}(\cdot)$ approaches this value. If $\beta < 1$, then the final controller K solves the VFSN performance feasibility problem.*

Proof: Omit due to page limit.

5 Example

We consider the VFSN control design for model of order 18, which is obtained in [10] through the closed loop identification of a 36th order structure. The VFSN uncertainty is used to characterize the modeling errors, computation errors and D/A-A/D errors. The model is of the form

$$\begin{aligned} \partial x &= A_p x + B_p(u + w + w_0) \\ z_0 &= C_0 x \\ z &= \begin{bmatrix} u \\ C_p x + D_p(u + w + w_0) \end{bmatrix} \\ y &= C_p x + D_p(u + w + w_0) + v + v_0 \end{aligned}$$

where $x \in \mathsf{R}^{18}$, the control output $u \in \mathsf{R}^3$ and the sensor output $y \in \mathsf{R}^3$. w, w_0, v and v_0 are independent white noise disturbances. $\mathcal{E}_\infty w_0^2 = W_0$, $\mathcal{E}_\infty v_0^2 = V_0$. w and v are the VFSN portions of sensor and actuator noises with variances satisfying

$$\begin{aligned} \mathcal{E}_\infty w_i^2 &= \sigma_i \, \mathcal{E}_\infty z_i^2, \; i = 1, 2, 3 \\ \mathcal{E}_\infty v_j^2 &= \sigma_j \, \mathcal{E}_\infty z_j^2, \; j = 4, 5, 6. \end{aligned}$$

We assume $0 \le \sigma_i \le 0.25$, $i = 1, 2, \cdots, 6$, i.e., the variance of signals is at least 4-times larger than the noise variance.

We choose $\lambda = 0.618$ in the Q–W iteration and the performance bound is set to be $\gamma = 0.1$. Figure 1 shows the plot of the largest eigenvalue of

$$\tilde{\mathsf{G}}\,\text{bdiag}(\frac{1}{\gamma}, 0.25I)$$

at each Q–W iteration. The iteration converges to $\mu_{\text{fsn}} = 0.4658$. For this μ_{fsn}, a 18th order controller is obtained.

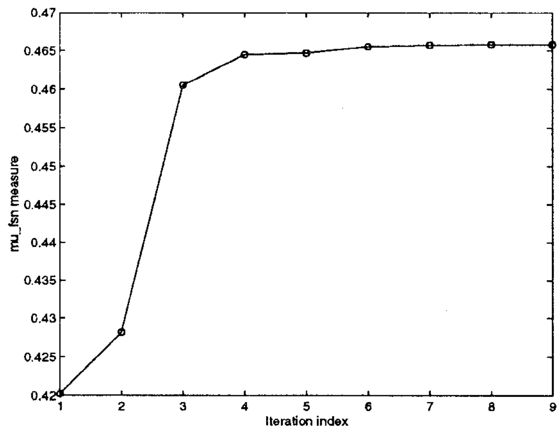

Figure 1: Iterative values for μ_{fsn} measure during Q–W iteration.

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