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Two Real Critical Constraints for Real Parameter Margin Computation (Part I: Theory and Applications)

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Abstract

A new approach to real parameter margin computation for dynamical systems with multilinearly uncertain parameters is presented. The concept of *two real critical constraints* is introduced. The proposed approach is essentially a frequency-sweeping approach which is based on a sufficient condition for checking for critical instability only in the corner directions of the parameter space hypercube.

1. Introduction

This paper is concerned with the problem of computing the structured singular values, μ, for uncertain dynamical systems [1]. In particular, the problem of computing real parameter margin, or real μ, is investigated, which is of much current research interest [2-7]. We exploit the concept of separating the real and imaginary parts of a characteristic polynomial equation. The resulting two equations will be referred to as the *two real critical constraints* and the vertex (or corner) property of each constraint equation will be utilized.

The paper is organized as follows. In Section 2, we introduce the *two real critical constraints*, the *two-constraint real* μ, and the *single-constraint real* μ. In Section 3, we show that the *single-constraint real* μ always reaches its value at a corner for multilinearly uncertain systems at a given frequency. We then present the main result: a sufficient condition for the critical instability to occur at a corner of the parameter space hypercube of multilinearly uncertain systems. In Section 4, we apply the results of Sections 2 and 3 to the problem of computing real parameter margins of different types of characteristic polynomial: an interval polynomial, a polytopic polynomial, and a polynomial with multilinearly uncertain parameters. In Part II of this paper [17], several examples in the literature are used to illustrate the proposed concept and approach.

2. Two Real Critical Constraints

Consider a characteristic polynomial $\phi(s;p)$ with the real uncertain parameter vector

$$p = (p_1, p_2, \cdots, p_\ell)$$

where

$$\underline{p}_i \le p_i \le \overline{p}_i, \quad i = 1, 2, \cdots, \ell$$

and $\underline{p}_i$ and $\overline{p}_i$ are, respectively, the prescribed lower and upper bounds of the ith element of the uncertain parameter vector p. The symbol $(p_1, p_2, \cdots, p_\ell)$ denotes a column vector in this paper; that is, $(p_1, p_2, \cdots, p_\ell) \equiv [p_1 \; p_2 \; \cdots \; p_\ell]^T$.

The normalized uncertain parameter vector $\delta \in \mathcal{D}$ is then defined as

$$\delta = (\delta_1, \delta_2, \cdots, \delta_\ell)$$

and the parameter space hypercube $\mathcal{D}$ is defined as

$$\mathcal{D} := \{\delta \; : \; -1 \le \delta_i \le 1, \;\; i = 1, \cdots, \ell\}$$

where

$$\delta_i = \frac{2(p_i - p_{0i})}{\overline{p}_i + \underline{p}_i} \tag{1}$$

and

$$p_{0i} = (\overline{p}_i + \underline{p}_i)/2, \quad i = 1, \cdots, \ell$$

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The nominal system of $\delta = 0$ is assumed to be asymptotically stable. We also define

$$\kappa\mathcal{D} := \{\delta \; : \; -\kappa \le \delta_i \le \kappa, \;\; i = 1, \cdots, \ell\}$$

where κ is a real positive number.

For a characteristic polynomial of the form $\phi(s;\delta)$, we have the following lemma.

Lemma 1: For any nth-order polynomial in s of the form $\phi(s;\delta)$ with uncertain parameter vector $\delta \in \mathcal{D}$, there exists an $n \times n$ rational matrix $M(s)$ and a diagonal matrix $\Delta \in X$ such that

$$\phi(s;\delta) = \phi(s;0)\det[I + M(s)\Delta] \tag{2}$$

where

$$X := \{\Delta \; : \; \Delta = \mathrm{diag}(\delta_i I_i), \;\; i = 1, \cdots, \ell\} \tag{3}$$

where I_i denotes an $m_i \times m_i$ identity matrix and $\sum_{i=1}^{\ell} m_i = n$

Proof: See Appendix.

According to Lemma 1, the critical stability constraint equation

$$\phi(j\omega;\delta) = 0 \tag{4}$$

simply becomes

$$\det[I + M(j\omega)\Delta] = 0 \tag{5}$$

since $\phi(j\omega;0) \ne 0$ for all ω.

Definition 1: The *real parameter robustness measure* $\kappa(\omega)$ and the *real structured singular value measure* $\mu(\omega)$ associated with the critical constraint equation (5) are defined as

$$\begin{aligned}
\kappa(\omega) &\equiv 1/\mu(\omega) \\
&:= \inf_{\Delta \in X} \{\kappa \; : \; \det[I + M(j\omega)\Delta] = 0, \; \bar{\sigma}(\Delta) \le \kappa\} \\
&:= \sup_{\Delta \in X} \{\kappa \; : \; \det[I + M(j\omega)\Delta] \ne 0, \; \bar{\sigma}(\Delta) \le \kappa\}
\end{aligned}$$

where X is the set of all repeated blocks defined as (3). The *real parameter margin* κ^* and the associated *real structured singular value* μ^* are then defined as

$$\kappa^* \equiv 1/\mu^* := \inf_{\omega} \kappa(\omega) \tag{6}$$

and the corresponding uncertain parameter vector is called the *critical parameter vector* and denoted by δ^*.

The critical stability constraint equation (4) or (5) is a complex constraint. We now exploit the idea of separating the real and imaginary parts of the constraint equation (4), as follows:

$$\mathrm{Re}[\phi(j\omega;\delta)] = f_1(\omega)\phi_1(\omega;\delta) = 0 \tag{7a}$$

$$\mathrm{Im}[\phi(j\omega;\delta)] = f_2(\omega)\phi_2(\omega;\delta) = 0 \tag{7b}$$

where $f_1(\omega)$ and $f_2(\omega)$ are polynomials which are independent of δ and $\phi_1(\omega;0) \ne 0$ and $\phi_2(\omega;0) \ne 0$ for all $\omega \ge 0$

According to Lemma 1, there exists a real rational matrix $M_1(\omega)$ and $M_2(\omega)$ such that

$$\phi_i(\omega;\delta) = \phi_i(\omega;0)\det[I + M_i(\omega)\Delta_i], \quad i = 1, 2 \tag{8}$$

Consequently, we have the following *two real critical constraints*

$$f_1(\omega)\det[I + M_1(\omega)\Delta_1] = 0 \tag{9a}$$

$$f_2(\omega)\det[I + M_2(\omega)\Delta_2] = 0 \tag{9b}$$

where

$$X_i := \{\Delta_i : \Delta_i = \mathrm{diag}(\delta_{ij} I_{ij}),\ \delta_{ij} \in \mathcal{R},\ j = 1, \cdots, \ell_i\},\ i = 1, 2 \tag{10}$$

and $\{\delta_{1j},\ j = 1, \cdots, \ell_1\}$ and $\{\delta_{2j},\ j = 1, \cdots, \ell_2\}$ are two subsets of $\{\delta_i, i = 1, \cdots, \ell\}$, and I_{ij} is an $m_{ij} \times m_{ij}$ identity matrix with $\sum_{j=1}^{\ell_i} m_{ij} = n_i,\ i = 1, 2$.

Polynomials with coefficients linearly dependent on and/or independent of uncertain parameters δ_i can be expressed as a form with rank-one matrices $M_1(\omega)$ and $M_2(\omega)$. This result is given as the following lemma.

Lemma 2: The critical stability constraint (4) of an interval polynomial or a polytopic polynomial can be expressed as two real critical constraints of the form (9) with rank-one matrices $M_1(\omega)$ and $M_2(\omega)$.

Proof: See Appendix.

Definition 2: A frequency at which the two real critical constraints (9) reduce to a single constraints is called the *degenerate frequency*.

Note that the real non-negative roots of the polynomials $f_1(\omega)$ and $f_2(\omega)$ of (9) are the degenerate frequencies. Degenerate frequencies cause isolated discontinuities in $\mu(\omega)$. These discontinuities include those denoted as type one and type two discontinuities in [9].

Definition 3: The *two-constraint real μ measure*, associated with the constraints (9), is defined as

$$1/\mu_{12}(\omega) := \inf_{\Delta \in X} \{\bar{\sigma}[\mathrm{diag}(\Delta_1, \Delta_2)] \ :\ \det[I + M_1 \Delta_1] = 0 \text{ and } \det[I + M_2 \Delta_2] = 0\} \tag{11}$$

The *single-constraint real μ measures*, $\mu_1(\omega)$ and $\mu_2(\omega)$, associated with each constraint in (9), are defined as:

$$1/\mu_1(\omega) := \inf_{\Delta_1 \in X_1} \{\bar{\sigma}(\Delta_1) : \det[I + M_1 \Delta_1] = 0\} \tag{12a}$$

$$1/\mu_2(\omega) := \inf_{\Delta_2 \in X_2} \{\bar{\sigma}(\Delta_2) : \det[I + M_2 \Delta_2] = 0\} \tag{12b}$$

The *real μ measure* of Definition 1 is related to the *two-constraint real μ measure* and the *single-constraint real μ measures* at each frequency ω, as follows:

$$\mu(\omega) = \begin{cases} \mu_{12}(\omega) & \text{if } f_1(\omega) \neq 0, f_2(\omega) \neq 0 \\ \mu_1(\omega) & \text{if } f_1(\omega) \neq 0, f_2(\omega) = 0 \\ \mu_2(\omega) & \text{if } f_1(\omega) = 0, f_2(\omega) \neq 0 \end{cases} \tag{13}$$

where $f_1(\omega)$ and $f_2(\omega)$ are the two polynomials defined in (7).

The following lemma provides sufficient conditions determining the *real μ^** using $\mu_{12}(\omega)$, $\mu_1(\omega)$ and $\mu_2(\omega)$.

Definition 4: Let $S = \{\delta_i,\ i = 1, \cdots, \ell\}$, S_1 and S_2 are two subsets of S and $S_1 \cup S_2 = S$. If $S_1 \cap S_2 \neq \emptyset$ we define the *restricted parameter vector* d in $S_1 \cap S_2$, as follows:

$$d = (d_1, \cdots, d_m);\ d_i \in S_1 \cap S_2,\ i = 1, \cdots, m \tag{14}$$

That is, d_i $(i = 1, \cdots, m)$ are some elements of $\delta = (\delta_1, \cdots, \delta_\ell)$ and $m \leq \ell$ is the number of elements in $S_1 \cap S_2$.

The restricted parameter vectors associated with μ_1 and μ_2 are denoted by d_{S_1} and d_{S_2}.

Lemma 3: Let $S = \{\delta_i,\ i = 1, \cdots, \ell\}$, S_1 and S_2 are two subsets of S with $S_1 \cup S_2 = S$. Then consider the following two cases: 1) $S_1 \cap S_2 = \emptyset$ and 2) $S_1 \cap S_2 \neq \emptyset$. By using the previous definitions of $\mu_1(\omega), \mu_2(\omega)$, and $\mu_{12}(\omega)$, we have the following results:

Case 1: If S_1 and S_2 are two distinct sets of the uncertain parameters, then the real μ^*, or the real parameter margin κ^*, can be found as:

$$1/\kappa^* \equiv \mu^* = \sup_\omega \mu_{12}(\omega) = \max\{\sup_\omega \mu_1(\omega), \sup_\omega \mu_2(\omega)\} \tag{15}$$

Case 2: If $S_1 \cap S_2 \neq \emptyset$ and at some critical frequencies ω_c,

$$\mu_1(\omega_c) = \mu_2(\omega_c)$$

and if the restricted parameter vectors in $S_1 \cap S_2$, associated with $\mu_1(\omega_c)$ and $\mu_2(\omega_c)$, become

$$d_{S_1} = d_{S_2} \tag{16}$$

then the real μ^* is

$$1/\kappa^* \equiv \mu^* = \sup_\omega \mu_{12}(\omega) = \max_{\omega_c} \mu_1(\omega_c) = \max_{\omega_c} \mu_2(\omega_c) \tag{17}$$

Proof: The proof of this lemma is rather trivial and is therefore omitted here.

3. Sufficient Conditions for Corner Property

A characteristic polynomial, which has coefficients affine with respect to each uncertain parameter δ_i, is called a *multilinearly uncertain polynomial*. A dynamical system with such characteristic polynomial is called a *multilinearly uncertain system* or a system with *multilinearly uncertain parameters*.

If a system is described by the critical stability constraint of the form (5) with

$$\Delta = \mathrm{diag}(\delta_1, \delta_2, \cdots, \delta_\ell) \tag{18}$$

$$m_i = 1 \quad \text{and} \quad i = 1, \cdots, \ell$$

then the system is a *multilinearly uncertain system*. However, not all multilinearly uncertain polynomials can be expressed in the form of (5) with (18); sometimes, Δ has repeated entries.

If the critical instability occurs at one of the corners of the parameter space hypercube, then the real parameter margin and the corresponding critical parameters can be easily determined using the following lemma.

Lemma 4: If the critical instability of the constraint (5) with possible repeated entries in Δ occurs at one of the corners of the parameter space hypercube, then

$$\kappa(\omega) = \{\max_{E \in \mathcal{E}} \rho[-EM(j\omega)]\}^{-1} \tag{19}$$

where $\rho(-EM)$ denotes the maximum real eigenvalue of $-EM$ and it is defined as zero if $-EM$ does not have real eigenvalues. Also, the *corner matrix*, denoted by E, is defined as

$$\mathcal{E} = \{E \ :\ E = \mathrm{diag}(e_i I_i),\ e_i = +1 \text{ or } -1,\ i = 1, \cdots, \ell\}$$

The real parameter margin κ^*, or real μ^*, is then determined as

$$\kappa^* \equiv 1/\mu^* := \inf_\omega \kappa(\omega)$$

The corresponding critical corner matrix E^* and critical corner vector e^* are, respectively, given by

$$E^* = \mathrm{diag}(e_i^* I_i) \tag{20}$$

$$e^* = (e_1^*, e_2^*, \cdots, e_\ell^*) \tag{21}$$

Furthermore, the critical parameter vector δ^* can be determined as

$$\delta^* = \kappa^* e^*$$

Proof: See [13]

Remark: If $E \in \mathcal{E}$, then $-E \in \mathcal{E}$ and $\lambda(EM) = -\lambda(-EM)$ where $\lambda(EM)$ denotes the eigenvalues of EM. Thus, $\kappa(\omega)$ defined in (19) is always positive real.

We now give the following sufficient condition for the corner property of a multilinearly uncertain system.

Lemma 5: If a multilinearly uncertain polynomial $\phi(j\omega; \delta)$ is always real-valued at some frequency ω, then the critical instability for this uncertain polynomial at that frequency occurs at one of the corners of the parameter space hypercube $\kappa^* \mathcal{D}$, where $\delta \in \kappa^* \mathcal{D}$, or $-\kappa^* \leq \delta_i \leq \kappa^*$ $(i = 1, 2, \cdots, \ell)$.

Proof: See [13,14]

Corollary of Lemma 5: In a multilinearly uncertain system, the single-constraint real μ defined as in Definition 3 must reach their values at one of the corner of the parameter space hypercube $\kappa^*\mathcal{D}$.

Theorem 1: At the degenerate frequencies defined as in Definition 2, the critical instability of a multilinearly uncertain system occurs at one of the corners of the parameter space hypercube.

Proof: At the degenerate frequencies, the two critical constraints (9) reduce to a single real valued linear or multilinear constraint, hence, we easily obtain this result from Lemma 5.

Corollary of Theorem 1: At $\omega = 0$, the critical instability occurs at one of the corners of the parameter space hypercube.

Proof: $M(j0)$ in the critical constraint (5) is a real-valued matrix. Hence, $\det[I + \Delta M(j0)]$ is a real-valued and $\omega = 0$ is one of the degenerate frequencies. From Theorem 1, we obtain this corollary.

Theorem 2: Consider the two real critical constraints (9) with multilinearly uncertain parameters.
Case 1 of Lemma 3: The critical instability occurs at one of the corners of the parameter space hypercube.
Case 2 of Lemma 3: If $\mu_1(\omega)$ and $\mu_2(\omega)$ plots intersect at some frequencies ω_c's and if the restricted parameters vector subject to $S_1 \cap S_2$ associated with $\mu_1(\omega_c)$ and $\mu_2(\omega_c)$ satisfy (16), then the critical instability occurs at one of the corners of the parameter space hypercube.

Proof: This theorem can be proved using Lemmas 3 and 5.

4. Applications

Interval Polynomial

Consider a family of real polynomials

$$\phi(s,p) = s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n \tag{22}$$

with interval coefficients described as

$$\underline{a}_i \leq a_i \leq \overline{a}_i, \quad i = 1,2,\cdots,n$$

The nominal values of a_i are

$$a_{0i} = (\overline{a}_i - \underline{a}_i)/2, \quad i = 1,2,\cdots,n$$

Using the parameter transformation of (1), we obtain the normalized parameters δ_i which are in $\mathcal{D}$.

A polynomial of the form (22) whose zeros lie on the open left-half s plane is called a Hurwitz polynomial. Kharitonov's theorem provides a simple way of checking whether a given interval polynomial is Hurwitz or not. In this section we use $\mu_i(\omega), i = 1,2$ to obtain the real parameter margin κ^* of an interval polynomial. An alternative proof of Kharitonov's theorem is also presented in this section.

From Lemma 2, an interval polynomial can be written as two real constraints with the rank-one matrices $M_i(\omega)$, $i = 1,2$. If n is an even integer, we have

$$\begin{aligned}
&M_i(\omega) = \frac{\alpha_i \beta_i^T(\omega)}{g_i(\omega)}, \quad i = 1,2\\
&\Delta_1 = \operatorname{diag}(\delta_2, \delta_4, \cdots, \delta_n), \quad \Delta_2 = \operatorname{diag}(\delta_1, \delta_3, ..., \delta_{n-1})\\
&\alpha_1 = \alpha_2 = [1,1,\cdots,1]^T \text{ is of } n/2\text{-dimension}\\
&\beta_1 = \beta_2 = [-\omega^{n-2}, \omega^{n-4}, \cdots, -\omega^2, (-1)^{n/2}]^T \text{ is of } n/2\text{-dimension}\\
&g_1(\omega) = \omega^n - a_{02}\omega^{n-2} + \cdots + (-1)^{n/2}a_{0n}\\
&g_2(\omega) = -a_{01}\omega^{n-2} + a_{03}\omega^{n-4} - \cdots + (-1)^{n/2}a_{0(n-1)}
\end{aligned}$$

Similar results can be obtained for the case of odd n.

Since $M_1(\omega)$ and $M_2(\omega)$ are rank-one matrices, we obtain the following theorem

Theorem 3: The μ_1 and μ_2 measures of an interval polynomial (22) reach their values at one of the corners of the parameter space hypercube and can be expressed as

$$\mu_i(\omega) = -\frac{\alpha_i^T E_i \beta_i(\omega)}{g_i(\omega)}, \quad i = 1,2 \tag{23}$$

where

$$E_i = -\operatorname{sgn}(g_i)\operatorname{diag}\{\operatorname{sgn}(\beta_{i1}, \operatorname{sgn}(\beta_{i2}), \cdots, 1\}, \quad i = 1,2 \tag{24}$$

and β_{ij} is the jth element of the column vector β_i, and $\operatorname{sgn}(\cdot)$ denotes the signum function. The real parameter margin is then obtained as

$$1/\kappa^* \equiv \mu^* \equiv \sup_\omega \mu_{12}(\omega) = \sup_\omega\{-\frac{\alpha_1^T E_1 \beta_1(\omega)}{g_1(\omega)}, -\frac{\alpha_1^T E_2 \beta_2(\omega)}{g_2(\omega)}\} \tag{25}$$

Proof: See Appendix

Kharitonov's Theorem

Kharitonov's theorem is proved here using Theorem 3. Without loss of generality, we only consider the case of even n.

The uncertain parameter set corresponding to μ_1 and μ_2 of an interval polynomial are disjoint (i.e., $S_1 \cap S_2 = \emptyset$), and those disjoint parameter sets have the same bound of κ^*. Consequently, from Theorem 3, we have either

$$\mu^* = -\frac{\alpha_1^T E_1 \beta_1(\omega_c)}{g_1(\omega_c)} = |-\frac{\alpha_1^T E_2 \beta_2(\omega_c)}{g_2(\omega_c)}| \tag{26}$$

or

$$\mu^* = -\frac{\alpha_1^T E_2 \beta_2(\omega_c)}{g_2(\omega_c)} = |-\frac{\alpha_1^T E_1 \beta_1(\omega_c)}{g_1(\omega_c)}| \tag{27}$$

where E_i are given in (24).

For the case with (26), the possible critical parameters are

$$\begin{aligned}
(\delta_2, \delta_4, \cdots, \delta_n) &= -\kappa^*[\operatorname{sgn}(\beta_{11}), \operatorname{sgn}(\beta_{12}), \cdots, 1]^T \operatorname{sgn}(g_1)\\
(\delta_1, \delta_3, \cdots, \delta_{n-1}) &= \pm\kappa^*[\operatorname{sgn}(\beta_{21}), \operatorname{sgn}(\beta_{22}), \cdots, 1]^T \operatorname{sgn}(g_2)
\end{aligned}$$

which give

$$\begin{aligned}
(\delta_2, \delta_4, \cdots, \delta_n) &= \kappa^*[-1,1,-1,\cdots,(-1)^{n/2+1}]^T \operatorname{sgn}(g_1)\\
(\delta_1, \delta_3, \cdots, \delta_{n-1}) &= \pm\kappa^*[-1,1,-1,\cdots,(-1)^{n/2+1}]^T \operatorname{sgn}(g_2)
\end{aligned}$$

For the case with (27), the possible critical parameters are

$$\begin{aligned}
(\delta_1, \delta_3, \cdots, \delta_{n-1}) &= -\kappa^*[\operatorname{sgn}(\beta_{21}), \operatorname{sgn}(\beta_{22}), \cdots, 1]^T \operatorname{sgn}(g_2)\\
(\delta_2, \delta_4, \cdots, \delta_n) &= \pm\kappa^*[\operatorname{sgn}(\beta_{11}), \operatorname{sgn}(\beta_{12}), \cdots, 1]^T \operatorname{sgn}(g_1)
\end{aligned}$$

which give

$$\begin{aligned}
(\delta_1, \delta_3, \cdots, \delta_{n-1}) &= \kappa^*[-1,1,-1,\cdots,(-1)^{n/2+1}]^T \operatorname{sgn}(g_2)\\
(\delta_2, \delta_4, \cdots, \delta_n) &= \pm\kappa^*[-1,1,-1,\cdots,(-1)^{n/2+1}]^T \operatorname{sgn}(g_1)
\end{aligned}$$

There are total sixteen combinations of possible critical parameters, but only four of them are different from each other. These four corner vectors for the possible critical instability are

$$\begin{aligned}
e &= [-1,-1,1,1,\cdots,(-1)^{n/2+1},(-1)^{n/2+1}]^T\\
e &= [-1,1,1,-1,\cdots,(-1)^{n/2},(-1)^{n/2+1}]^T\\
e &= [1,-1,-1,1,\cdots,(-1)^{n/2+1},(-1)^{n/2}]^T\\
e &= [1,1,-1,-1,\cdots,(-1)^{n/2},(-1)^{n/2}]^T
\end{aligned}$$

which are, in fact, Kharitonov's four corners.

Polytopic Polynomial

Consider a polynomial whose coefficients depend linearly on the perturbation parameter vector $\delta \in \kappa^*\mathcal{D}$

$$\phi(s;\delta) = s^n + \sum_{i=1}^{n} a_i(\delta)s^{n-i} \tag{28}$$

where

$$a_i(\delta) = a_{0i} + \sum_{j=1}^{\ell} a_{ij}\delta_j, \quad a_{ij} \text{ are constants}$$

From Lemma 2, the critical constraints can be written as (9) with rank-one matrices $M_1(\omega)$ and $M_2(\omega)$ such that

$$M_i(\omega) = \alpha_i(\omega)\beta_i^T(\omega), \quad i = 1,2$$

Consequently, we obtain the following result

Theorem 4: The *two single-constraint real μ measures* of a polytopic polynomial of (28) will reach their values at one of the corners of the parameter space hypercube and can be expressed as

$$\mu_i(\omega) = -\alpha_i^T(\omega)E_i\beta_i(\omega), \quad i = 1,2 \tag{29}$$

where

$$E_i = -\operatorname{diag}\{\operatorname{sgn}(\alpha_{i1}\beta_{i1}), \operatorname{sgn}(\alpha_{i2}\beta_{i2}), \cdots, \operatorname{sgn}(\alpha_{i\ell_i}\beta_{i\ell_i})\}$$

for $i = 1,2$, and α_{ij} and β_{ij} are, respectively, the jth elements of the column vectors α_i and β_i.

If $\mu_1(\omega)$ and $\mu_2(\omega)$ intersect at some frequencies and at those frequencies the overlapped part of uncertain parameters are of the same values in the corner matrices for both $\mu_1(\omega)$ and $\mu_2(\omega)$, then the critical instability occurs at one of the corners of parameter space hypercube.

Proof: The first part of this theorem is an extension of Theorem 3. The second part is an obvious application of Theorem 2.

Multilinearly Uncertain Polynomial

For a general case of multilinearly uncertain polynomial, from Corollary of Lemma 5, we know $\mu_1(\omega)$ and $\mu_2(\omega)$ will reach their values at one of the corners of the parameter space hypercube $\mathcal{D}$ at any frequency ω. Hence we have

Theorem 5: The *two single-constraint real μ measures* will reach their values at one of the corners of the parameter space hypercube and can be expressed as

$$\mu_i(\omega) = \max_{E_i \in \mathcal{E}_i} \rho[-E_i M_i(\omega)], \ i = 1,2$$

where

$$\mathcal{E}_i := \{E_i \ : \ E_i = \operatorname{diag}(e_j I_j), \ e_j = +1 \text{ or } -1 \ \forall j\}$$

$\rho(\cdot)$ denotes the maximum real eigenvalue of a matrix. If $\mu_1(\omega)$ and $\mu_2(\omega)$ intersect at some frequencies and at those frequencies the overlapped part of uncertain parameters are of the same values in the critical corner matrices for both $\mu_1(\omega)$ and $\mu_2(\omega)$, then the critical stability occurs at one of the corners.

Proof: The first part comes directly from Lemma 4. The second part is the direct application of Theorem 2.

5. Examples

See [17]

6. Conclusions

The concept of *two real critical constraints* was exploited for the real parameter margin computation for dynamical systems with multilinearly uncertain parameters. The proposed approach is based on a sufficient condition for checking for critical instability only in the corner directions of the parameter space hypercube.

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Appendix: Proofs of Lemmas and Theorems

Proof of Lemma 1: Let us first prove if a given polynomial $\phi(s;\delta)$ can be written as $\phi(s;\delta) = f(s;\delta)\det[I + M(s)\Delta]$ where $f(s;\delta) = \phi(s;0)$ or not.

For a given rational square matrix $[I + M(s)\Delta(\delta)]$ with the elements of δ vector appearing in the numerators of each element of $[I + M(s)\Delta(\delta)]$, there exist polynomials $\phi(s;\delta)$, $\delta \in \mathcal{D}$ and $\psi(s)$ such that

$$\det[I + M(s)\Delta(\delta)] = \frac{\phi(s;\delta)}{\psi(s)}$$

because of the property of Smith-McMillan form (e.g., see Theorem 2.3 in [16]) of $[I + M(s)\Delta(\delta)]$.

Since $\Delta(\delta) = \operatorname{diag}(\delta_i)$, $\Delta(0) = 0$. Consequently, we have

$$1 = \det[I] = \frac{\phi(s;0)}{\psi(s)} \Rightarrow \psi(s) = \phi(s;0)$$

Finally we have

$$\det[I + M(s)\Delta] = \frac{\phi(s;\delta)}{\phi(s;0)} \tag{30}$$

The existence of $M(j\omega)$ and Δ is obvious. But the actual determination of $M(j\omega)$ and Δ is not straightforward. Without loss of generality, we use the following example to show how to form M and Δ for a given $\phi(s;\delta)$.

Consider a polynomial with coefficients which are multilinearly dependent on the four uncertain parameters δ_i:

$$\begin{aligned}\phi(s;\delta) = {} & a_0 + a_1\delta_1 + a_2\delta_2 + a_3\delta_3 + a_4\delta_4 \\ & + b_1\delta_1\delta_2 + b_2(\delta_1 + \delta_2)\delta_3 + b_3(\delta_1 + \delta_2)\delta_4\end{aligned}$$

where a_i and b_i are functions of s.

We first transform this given polynomial into the determinant form of a 2×2 matrix, as follows:

$$\phi(s;\delta) = \det[h_0 + h_1\delta_1 + h_2\delta_2 + h_3\delta_3 + h_4\delta_4 + h_5\delta_1\delta_2]$$

where h_i, $(i = 0, \cdots, 5)$ are 2×2 matrices, as follows

$$h_0 = \begin{bmatrix} 0 & a_0 \\ -1 & 0 \end{bmatrix},\ h_1 = \begin{bmatrix} 0 & a_1 \\ 0 & 1 \end{bmatrix},\ h_2 = \begin{bmatrix} 0 & a_2 \\ 0 & 1 \end{bmatrix}$$

$$h_3 = \begin{bmatrix} b_2 & a_3 \\ 0 & 0 \end{bmatrix},\ h_4 = \begin{bmatrix} b_3 & a_4 \\ 0 & 0 \end{bmatrix},\ h_5 = \begin{bmatrix} 0 & b_1 \\ 0 & 0 \end{bmatrix}$$

In order to change the term $h_5\delta_1\delta_2$ into linear fractional form, we further express the determinant as the determinant of a 4×4 matrix

$$\phi(j\omega;\delta) = \det[H_0 + H_1\delta_1 + H_2\delta_2 + H_3\delta_3 + H_4\delta_4] \tag{31}$$

where $H_i (i = 0, \cdots, 4)$ are 4×4 matrices

$$H_0 = \begin{bmatrix} h_0 & 0 \\ 0 & I_{2\times2} \end{bmatrix},\ H_1 = \begin{bmatrix} h_1 & h_5 \\ 0 & 0 \end{bmatrix}$$

$$H_2 = \begin{bmatrix} h_2 & 0_{2\times2} \\ -I_{2\times2} & 0_{2\times2} \end{bmatrix},\ H_3 = \begin{bmatrix} h_3 & 0_{2\times2} \\ 0_{2\times2} & 0_{2\times2} \end{bmatrix}$$

$$H_4 = \begin{bmatrix} h_4 & 0_{2\times2} \\ 0_{2\times2} & 0_{2\times2} \end{bmatrix}$$

Taking the singular value decomposition of H_i, $(i = 1\cdots)$, we obtain

$$H_i = L_i R_i$$

with $\mathrm{rank}[L_i] = \mathrm{rank}[R_i] = r_i$.

Then (31) can be written as

$$\phi(s;\delta) = \det[H_0]\det[I + M(s)\Delta] \tag{32}$$

where

$$M = RH_0^{-1}L$$

and

$$\begin{aligned}R &= [R_1^T, R_2^T, \cdots, R_5^T]^T \\ L &= [L_1, L_2, \cdots, L_5] \\ \Delta &= \mathrm{diag}(\delta_i I_{r_i})\end{aligned}$$

Finally we obtain

$$R = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ b_2 & a_3 & 0 & 0 \\ b_3 & a_4 & 0 & 0 \end{bmatrix}$$

$$L = \begin{bmatrix} a_1 & b_1 & 0 & a_2 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \end{bmatrix}$$

and

$$\Delta = \mathrm{diag}(\delta_1, \delta_1, \delta_2, \delta_2, \delta_3, \delta_4)$$

This form of M is not necessarily unique.

We can show that (32) can be expressed as (30) since

$$\det[H_0] = \det[h_0] = \det\begin{bmatrix} 0 & a_0 \\ -1 & 0 \end{bmatrix} = a_0 = \phi(s,0)$$

Proof of Lemma 2:

Consider the constraint with the linearly-dependent and/or linearly-independent uncertain parameters of the form (4).

$$\phi(s;\delta) = \phi(s;0) + \sum_{i=1}^{\ell} a_i(s)\delta_i = 0$$

where $\delta \in \mathcal{D}$ denotes the uncertain parameter vector and $\mathcal{D}$ denotes the parameter space hypercube. This polynomial equation can be separated as follows:

$$\mathrm{Re}[\phi(s;0)] + \sum_{i=1}^{\ell} \mathrm{Re}[a_i]\delta_i = 0 \tag{33}$$

$$\mathrm{Im}[\phi(s;0)] + \sum_{i=1}^{\ell} \mathrm{Im}[a_i]\delta_i = 0 \tag{34}$$

Equation (33) can be rewritten as

$$\begin{aligned}0 &= \det\begin{bmatrix} \mathrm{Re}[\phi(s;0)] + \sum_{i=1}^{\ell}\mathrm{Re}[a_i]\delta_i & 0 \\ 0 & 1 \end{bmatrix} \\ &= \mathrm{Re}[\phi(s;0)]\det[I + \tilde{M}\Delta]\end{aligned}$$

where

$$\Delta = \mathrm{diag}(\delta_1, \delta_2, ..., \delta_\ell)$$

$$\tilde{M} = \frac{1}{\mathrm{Re}[\phi(s;0)]}\begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}\begin{bmatrix} \mathrm{Re}[a_1] & \ldots & \mathrm{Re}[a_\ell] \end{bmatrix}$$

Note that $\tilde{M}$ is a rank-one matrix. Similar result can be obtained for the other constraint (34).

Proof of Theorem 3:

From Corollary of Theorem 2 we know $\mu_1(\omega)$ and $\mu_2(\omega)$ will reach their values at one of the corners of the parameter space hypercube. Hence, according to Lemma 4, we have

$$\mu_i(\omega) = \max_{E_i \in \mathcal{E}_i} \rho[-E_i M_i(j\omega)], \quad i = 1, 2$$

Since

$$\begin{aligned}\det[\lambda I + \frac{E\alpha_i(\omega)\beta_i^T(\omega)}{g_i(\omega)}] &= \lambda + \frac{\beta_i^T(\omega)E\alpha_i(\omega)}{g_i(\omega)} \\ &= \lambda + \frac{\alpha_i^T(\omega)E\beta_i(\omega)}{g_i(\omega)}\end{aligned}$$

we have

$$\mu_i(\omega) = \max_{E\in\mathcal{E}}\{-\frac{\alpha_i^T(\omega)E\beta_i(\omega)}{g_i(\omega)}\}$$

Let

$$E_i = -\mathrm{sgn}(g_i)\,\mathrm{diag}\{\mathrm{sgn}(\beta_{i1}), \mathrm{sgn}(\beta_{i2}), ..., 1\}$$

then $\mu_i(\omega)$ will reach their maximum values.