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Two Real Critical Constraints for Real Parameter Margin Computation (Part II: Examples)

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Example 1: Vicino and Tesi [10]

Consider a polynomial with linearly-dependent uncertain parameters δ_1 and δ_2:

$$\phi(s;\delta_1,\delta_2) = s^4 + (\delta_2+3)s^3 + (\delta_1+5.5)s^2 + (\delta_1+\delta_2+4.5)s + 3\delta_1 - \delta_2 + 5.5$$

The two real critical constraints can be found as

$$\mathrm{Re}\{\phi(j\omega;\delta)\} = 0 \rightarrow f_1(\omega)\det[I_1 + \Delta_1 M_1(\omega)] = 0$$
$$\mathrm{Im}\{\phi(j\omega;\delta)\} = 0 \rightarrow f_2(\omega)\det[I_2 + \Delta_2 M_2(\omega)] = 0$$

where

$$f_1(\omega) = 1,\quad f_2(\omega) = \omega,\quad \Delta_1 = \Delta_2 = \mathrm{diag}(\delta_1,\delta_2)$$

$$M_1(\omega) = \frac{1}{\omega^4 - 5.5\omega^2 + 5.5}\begin{bmatrix}1\\1\end{bmatrix}[3-\omega^2,\quad -1]$$
$$M_2(\omega) = \frac{1}{-3\omega^2 + 4.5}\begin{bmatrix}1\\1\end{bmatrix}[1,\quad 1-\omega^2]$$

And we obtain

$$\mu_1(\omega) = \frac{|3-\omega^2|+1}{|\omega^4 - 5.5\omega^2 + 5.5|}$$
$$\mu_2(\omega) = \frac{|1-\omega^2|+1}{|-3\omega^2+4.5|}$$

The critical corner matrices of $\mu_1(\omega)$ and $\mu_2(\omega)$ are

$$E_1 = -\mathrm{diag}\{\mathrm{sgn}(3-\omega^2),\ \mathrm{sgn}(-1)\}\mathrm{sgn}(\omega^4 - 5.5\omega^2 + 5.5)$$
$$E_2 = -\mathrm{diag}\{\mathrm{sgn}(1),\ \mathrm{sgn}(1-\omega^2)\}\mathrm{sgn}(-3\omega^2 + 4.5)$$

At the degenerate frequency of $\omega = 0$, we have

$$\mu(0) = \mu_1(0) = 1.375$$

By solving $\mu_1(\omega_c) = \mu_2(\omega_c)$ for ω_c, we find four critical frequencies. The corner matrices of $\mu_1(\omega)$ and $\mu_2(\omega)$ are the same only at $\omega_c = 1.4142$, and they are $E_1 = E_2 = \mathrm{diag}(1,-1)$. Since $\mu_1(1.4142) < \mu(0)$, the critical instability occurs at $\omega_c = 1.4142$. The critical corner matrix is $E^* = E_1 = E_2 = \mathrm{diag}(1,-1)$ and the critical corner vector is $e^* = (1,-1)$.

The real parameter margin κ^* becomes

$$\kappa^* = 1/\mu_{12} = 1/\mu_1(\omega_c) = 1/\mu_2(\omega_c 2) = 0.75$$

where $\omega_c = 1.4142$ and the critical parameter values are:

$$(\delta_1,\delta_2) = \kappa^* e^* = \kappa^*(1,-1) = (0.75, -0.75)$$

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Example 2: De Gaston and Safonov [3]

Consider a feedback control system consisting of a plant transfer function $G(s)$ and a compensator $K(s)$ given by

$$G(s) = \frac{p_1}{s(s+p_2)(s+p_3)};\quad K(s) = \frac{s+2}{s+10}$$

The uncertain parameters are described by:

$$\begin{aligned} p_1 &= 800(1+\epsilon_1), & |\epsilon_1| \le 0.1\\ p_2 &= 4+\epsilon_2, & |\epsilon_2| \le 0.2\\ p_3 &= 6+\epsilon_3, & |\epsilon_3| \le 0.3 \end{aligned}$$

The closed-loop characteristic polynomial is

$$\phi(s;\epsilon_1,\epsilon_2,\epsilon_3) = s^4 + a_1 s^3 + a_2 s^2 + a_3 s + a_4$$

where

$$\begin{aligned} a_1 &= 20 + \epsilon_2 + \epsilon_3\\ a_2 &= 124 + 16\epsilon_2 + 14\epsilon_3 + \epsilon_2\epsilon_3\\ a_3 &= 1040 + 800\epsilon_1 + 60\epsilon_2 + 40\epsilon_3 + 10\epsilon_2\epsilon_3\\ a_4 &= 1600(1+\epsilon_1) \end{aligned}$$

The uncertain parameters are normalized as

$$\delta_1 = \epsilon_1/0.1,\quad \delta_2 = \epsilon_2/0.2,\quad \delta_3 = \epsilon_3/0.3$$

The two real critical constraints can then be obtained as:

$$\mathrm{Re}\{\phi(j\omega;\delta)\} = 0 \rightarrow f_1(\omega)\det[I_1 + \Delta_1 M_1(\omega)] = 0$$
$$\mathrm{Im}\{\phi(j\omega;\delta)\} = 0 \rightarrow f_2(\omega)\det[I_2 + \Delta_2 M_2(\omega)] = 0$$

where

$$\begin{aligned} &f_1(\omega) = 1,\quad f_2(\omega) = \omega\\ &\Delta_1 = \Delta_2 = \Delta = \mathrm{diag}(\delta_1,\delta_2,\delta_3)\\ &M_i(\omega) = R_i(\omega)A_i(\omega)L_i(\omega),\quad i = 1,2 \end{aligned}$$

and

$$R_1(\omega) = \begin{bmatrix} 1 & 0\\ -3.2\omega^2 & 1\\ 1 & 0 \end{bmatrix}$$

$$R_2(\omega) = \begin{bmatrix} 1 & 0\\ 12 - 0.2\omega^2 & -1\\ 1 & 0 \end{bmatrix}$$

$$L_1(\omega) = \begin{bmatrix} 160 & 1 & -4.2\omega^2\\ 0 & 0 & 0.06\omega^2 \end{bmatrix}$$

$$L_2(\omega) = \begin{bmatrix} 80 & 1 & 12 - 0.3\omega^2\\ 0 & 0 & 0.6 \end{bmatrix}$$

$$A_1(\omega) = \begin{bmatrix} (1600 - 124\omega^2 + \omega^4)^{-1} & 0\\ 0 & 1 \end{bmatrix}$$

$$A_2(\omega) = \begin{bmatrix} (1040 - 20\omega^2)^{-1} & 0\\ 0 & 1 \end{bmatrix}$$

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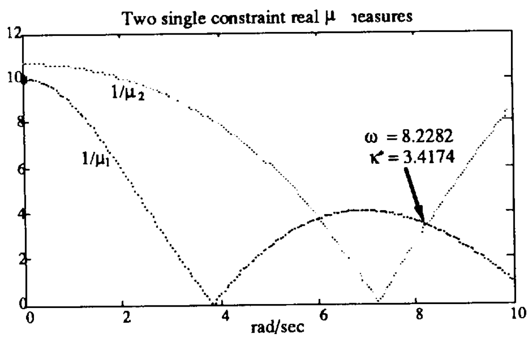

Figure 1: Example 2.

As shown in Fig. 1, plots of $\mu_1(\omega)$ and $\mu_2(\omega)$ versus ω intersect at two frequencies. We further find that only at $\omega_c = 8.2282$, the corner directions for both $\mu_1(\omega)$ and $\mu_2(\omega)$ are the same. Thus the critical corner matrices are

$$E_1 = E_2 = \mathrm{diag}(1, -1, -1)$$

and the real parameter margin is found as

$$\kappa^* = 1/\mu^* = 1/\mu_1(\omega_c) = 1/\mu_2(\omega_c) = 3.4174$$

where $\omega_c = 8.2282$. The corresponding critical corner vector is $e = (1, -1, -1)$ and the critical parameter values are

$$(p_1, p_2, p_3) = (1073.36, 3.3165, 4.9748)$$

Example 3: Ackermann's Multilinear Polynomial [15,8]

Consider a characteristic polynomial with ℓ uncertain parameters p_i given by:

$$\phi(s;p) = \ell(\ell-1) + r^2 + 2(\ell+1)\sum_{i=1}^{\ell} p_i + 2\sum_{i<j}^{\ell} p_i p_j + (\ell + \sum_{i=1}^{\ell} p_i)s + (\ell + \sum_{i=1}^{\ell} p_i)s^2 + s^3$$

Here we consider a case of

$$\ell = 4, \quad r = 0.5, \quad p_{0i} = 4\ ; \quad (i = 1, ..., 4)$$

and let

$$p_i = p_{0i} + \delta_i\ ; \quad (i = 1, ..., 4)$$

The two real critical constraints can be found as follows:

$$\mathrm{Re}\{\phi(j\omega;\delta)\} = 0 \rightarrow f_1(\omega)\det[I_1 + \Delta_1 M_1(\omega)] = 0$$
$$\mathrm{Im}\{\phi(j\omega;\delta)\} = 0 \rightarrow f_2(\omega)\det[I_2 + \Delta_2 M_2(\omega)] = 0$$

where

$$f_1(\omega) = 1, \quad f_2(\omega) = \omega$$
$$\Delta_1 = \mathrm{diag}(\delta_1, \delta_2, \delta_2, \delta_3, \delta_3, \delta_3, \delta_3, \delta_4)$$
$$\Delta_2 = \mathrm{diag}(\delta_1, \delta_2, \delta_3, \delta_4)$$

and

$$M_1(\omega) = R_1(\omega)A_1^{-1}(\omega)L_1(\omega)$$

$$M_2(\omega) = \frac{\alpha\alpha^T}{p_{01} + p_{02} + p_{03} + p_{04} + 4 - \omega^2}$$

$$R_1(\omega) = \begin{bmatrix} R_{11} & 0 \\ I_{4\times4} & 0 \\ R_{31} & R_{32} \end{bmatrix}$$

where

$$R_{11} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$R_{31} = [10 - \omega^2,\ 2,\ 2,\ 0]$$
$$R_{32} = [2,\ 0,\ 0,\ 0]$$

$$A_1(\omega) = \begin{bmatrix} A_{11} & A_{12} \\ -p_{03}I_{4\times4} & I_{4\times4} \end{bmatrix}$$

$$A_{11} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & 0 \\ -p_{01} & 1 & 0 & 0 \\ -p_{02} & 0 & 1 & 0 \\ 0 & -p_{02} & 0 & 1 \end{bmatrix}$$

$$a_{11} = 12 + r^2 - 4\omega^2 + (10 - \omega^2)(p_{01} + p_{02} + p_{03} + p_{04})$$
$$a_{12} = 2(p_{02} + p_{03} + p_{04})$$
$$a_{13} = 2(p_{03} + p_{04})$$

$$A_{12} = \begin{bmatrix} 2p_{04} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$L_1(\omega) = \begin{bmatrix} L_{11} & 1 \\ I_{7\times7} & 0 \end{bmatrix}$$

$$L_{11} = [10 - \omega^2,\ 10 - \omega^2,\ 2,\ 10 - \omega^2,\ 2,\ 2,\ 0]$$
$$\alpha = [1,\ 1,\ 1,\ 1]^T$$

Since $M_2(\omega)$ is a rank-one matrix, we have

$$\mu_2(\omega) = \frac{\alpha^T\alpha}{|p_{01} + p_{02} + p_{03} + p_{04} + 4 - \omega^2|}$$

By Lemma 4, $\mu_1(\omega)$ can be found by checking corners, as follows:

$$\mu_1(\omega) = \max_{E_1 \in \mathcal{E}_1} \rho[-E_1 M_1(\omega)]$$

As shown in Fig. 2, we find that the $\mu_1(\omega)$ and $\mu_2(\omega)$ plots intersect at two frequencies:

$$\omega_{c1} = 2.6458 \text{ and } \omega_{c2} = 3.0000$$

At these frequencies, $\mu_1(\omega_c)$ and $\mu_2(\omega_c)$ have the following corner matrices

$$E_1 = \mathrm{diag}(e_1, e_2, e_2, e_3, e_3, e_3, e_3, e_4)$$
$$E_2 = \mathrm{diag}(e_1, e_2, e_3, e_4)$$
$$e_1 = e_2 = e_3 = e_4 = 1$$

Then we have

$$\mu(\omega_{c1}) = \mu_{12}(\omega_{c1}) = \mu_2(\omega_{c1}) = 0.3636$$
$$\mu(\omega_{c2}) = \mu_{12}(\omega_{c2}) = \mu_2(\omega_{c2}) = 0.3077$$

The real parameter margin is found as

$$\kappa^* = 1/\mu^* = \min_{\omega_c}\{1/\mu_{12}(\omega_c)\} = 2.75$$

and the critical parameter values are

$$(p_1, p_2, p_3, p_4) = (1.25, 1.25, 1.25, 1.25)$$

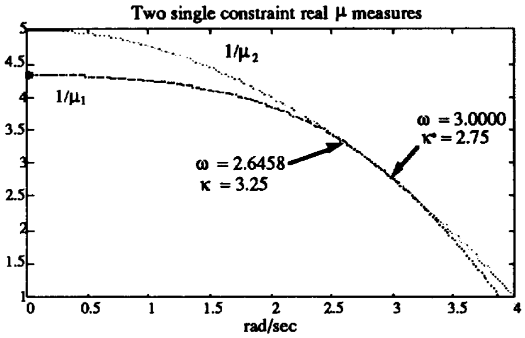

Figure 2: Example 3

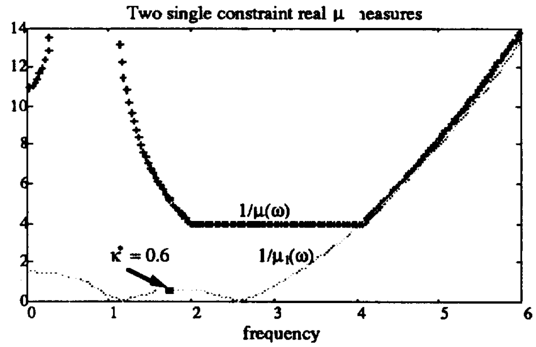

Figure 3: Example 5

Example 4: Chang et al. [6]

Consider the closed-loop characteristic polynomial of the second example of [6]:

$$\phi(s;\delta_1,\delta_2,\delta_3) = s^4 + a_1 s^3 + a_2 s^2 + a_3 s + a_4 \qquad (1)$$

where

$$\begin{aligned} a_1 &= 10.4 - 0.3\delta_1 - 0.3\delta_2 \\ a_2 &= 38.14 - 2.31\delta_1 - 2.91\delta_2 + 0.45\delta_3 + 0.09\delta_1\delta_2 \\ a_3 &= 58.12 - 5.97\delta_1 - 8.28\delta_2 + 1.74\delta_3 + 0.63\delta_1\delta_2 \\ &\quad - 0.135\delta_2\delta_3 \\ a_4 &= 31.16 - 5.22\delta_1 - 6.84\delta_2 + 0.48\delta_3 + 1.08\delta_1\delta_2 \\ &\quad - 0.27\delta_2\delta_3 \end{aligned}$$

One of the critical constraints has the form of

$$\begin{aligned} \Delta_1 &= \mathrm{diag}(\delta_1,\delta_2,\delta_3) \\ M_1(\omega) &= R_1(\omega)A_1(\omega)L_1(\omega) \end{aligned}$$

and

$$R_1(\omega) = \begin{bmatrix} 2.31\omega^2 - 5.22 & 0.09\omega^2 - 1.08 \\ 1 & 0 \\ 0.48 - 0.45\omega^2 & 0.27 \end{bmatrix}$$

$$L_1(\omega) = \begin{bmatrix} 1 & 2.91\omega^2 - 6.84 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$A_1(\omega) = \begin{bmatrix} (\omega^4 - 38.14\omega^2 + 31.36)^{-1} & 0 \\ 0 & 1 \end{bmatrix}$$

Hence the critical parameters will attain their values at the corner according to Corollary of Theorem 1. The two constraint real μ or real μ can be found as

$$\mu^* = \mu_{12}(\omega_c) = \mu_1(\omega_c) = \max_{E\in\mathcal{E}} \rho[-EM_1(\omega_c)] = 0.2755$$

where $\omega_c = 0$. The real parameter margin is

$$\kappa^* = 1/\mu^* = 3.6297$$

and the corresponding critical corner matrix is

$$E^* = \mathrm{diag}(1,1,1)$$

For the nominal values of $\delta_i = 0$ $(i = 1,2,3)$, the corresponding critical parameter values are

$$(\delta_1,\delta_2,\delta_3) = (3.6297, 3.6297, 3.6297)$$

Example 5: Barmish et al. [12]

Consider a polynomial:

$$\begin{aligned} \phi(s;\delta_1,\delta_2) = s^4 &+ (4-\delta_2)s^3 + (8-2\delta_1)s^2 \\ &+ (12 - 3\delta_2)s + (9 - \delta_1 - 5\delta_2) \end{aligned}$$

We use this example to show that the discontinuity in the real μ measure occurs at the degenerate frequencies and to further demonstrate that the real μ can be determined using one of the single-constraint real μ.

The two real critical constraints are

$$\mathrm{Re}\{\phi(j\omega)\} = 0 \rightarrow f_1(\omega)\det[I + \Delta_1 M_1(\omega)] = 0$$

$$\mathrm{Im}\{\phi(j\omega)\} = 0 \rightarrow f_2(\omega)\det[I + \Delta_2 M_2(\omega)] = 0$$

where

$$\begin{aligned} &\Delta_1 = \mathrm{diag}(\delta_1,\delta_2), \;\; \Delta_2 = \delta_2 \\ &f_1(\omega) = 1, \;\; f_2(\omega) = 4(3-\omega^2)\omega \\ &M_1(\omega) = \frac{1}{9 - 8\omega^2 + \omega^4}\begin{bmatrix} 2\omega^2 - 1 & -5 \\ 2\omega^2 - 1 & -5 \end{bmatrix} \\ &M_2(\omega) = -1/4 \end{aligned}$$

At $\omega = \sqrt{3}$, the two real critical constraints reduce to one real critical constraint; i.e., $\omega = \sqrt{3}$ is the *degenerate frequency*. Since there are two constraints when $\omega \neq \sqrt{3}$, the two constraint real μ measure $\mu_{12}(\omega)$ becomes discontinuous at $\omega = \sqrt{3}$. As shown in [12], the worst case occurs at $\omega = \sqrt{3}$ and it should occur at a corner. Thus the two single constraint real μ measures are

$$\mu_1(\omega) = \frac{|2\omega^2 - 1| + 5}{|9 - 8\omega^2 + \omega^4|}, \;\; \mu_2(\omega) = 0.25$$

See Fig. 3 for $1/\mu_1(\omega)$ plot. The real μ^* or the real parameter margin κ^* is then found as

$$\mu^* = 1/\kappa^* = \mu_1(\sqrt{3}) = 5/3$$

References

See references given in Part I.