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Nonlinear Quaternion Feedback Control for Spacecraft via Angular Velocity Shaping

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Abstract

A nonlinear feedback control logic is developed for a spacecraft attitude control through angular velocity shaping in time domain. The proposed control logic guarantees slew rate constraint and can tolerate uncertainties in the inertia matrix.

1. Introduction

In order to prevent actuator saturation, energy consumption and excitation of the structural vibration modes, in spacecraft attitude control, it is required to constrain rate and actuator variables. In [1] a feedback control logic based on multi-layer saturation structure is developed such that those constraints are satisfied, but this multilayer saturation controller can not handle parameter uncertainties; requires opposite reactions to accelerate and decelerate the spacecraft, which may be some source of excitation of flexible modes. On the other hand, how to choose some of the design parameter is not clear. As an alternate approach, the controller proposed in this paper try to be a complementary to controller proposed in [1].

This paper is organized as follows. In section 2, the problem formulation is given. In section 3, the desired angular velocity to be shaped is obtained. An angular velocity shaping problem is formed in section 4 and the final control logic is derived from angular velocity tracking condition. The robustness issue is discussed. We discuss the time response performance of the closed system in section 5. The design procedure is summarized in section 6. An illustrative example is given in section 7. Some conclusions are given in section 8.

2. Problem Formulation

The dynamics of a rigid-body spacecraft are described by Euler's dynamical equations and the kinematic equations. Euler's equations express the rotational motion of the rigid body spacecraft about the body-fixed axes with their origin at the center of mass. The equations that follow are associated with the general case in which the body-fixed control axes do not coincide with the principal axes of inertia (i.e., the J matrix is not necessarily diagonal matrix)

$$J\dot{\omega} = \Omega J\omega + u \tag{1}$$

where $\omega = [\omega_1,\ \omega_2,\ \omega_3]^T$ is the angular velocity vector, $u = [u_1,\ u_2,\ u_3]^T$ the control vector, J the inertia matrix, and Ω a skew-symmetric matrix defined by

$$\Omega = -\begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}$$

The subscripts 1, 2 and 3 denote the body-fixed axes. It is assumed that the angular velocity components along the body-fixed control axes are measured by rate gyros.

Since quaternion are well suited for onboard real-time computation, spacecraft orientation is now commonly described in terms of quaternion (e.g., HEAO, Space Shuttle, and Galileo) [2, 3]. The quaternion defines the rigid-body attitude as a Euler-axis rotation. The vector part of the quaternion indicates the direction of the Euler axis. The 4-th scale part of the quaternion is related to the rotation angle about the Euler axis [3]. The quaternion $\hat{q}_i$, $i = 1,2,3,4$ are defined as

$$\hat{q} = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix} = \begin{bmatrix} c_1 \sin(\phi/2) \\ c_2 \sin(\phi/2) \\ c_3 \sin(\phi/2) \\ \cos(\phi/2) \end{bmatrix} \tag{2}$$

where ϕ is the magnitude of the Euler axis rotation and $(c_1,\ c_2,\ c_3)$ are the direction cosines of the Euler axis relative to a reference frame. It is obvious that $\hat{q}^T\hat{q} = 1$

The quaternion kinematic differential equation is described by

$$\dot{q} = \frac{1}{2}\Omega q + \frac{1}{2}q_4\omega \tag{3a}$$

$$\dot{q}_4 = -\frac{1}{2}\omega^T q \tag{3b}$$

where $q = [q_1, q_2, q_3]^T$ is the vector part of quaternion $\hat{q}$.

Obviously, the dynamics combining (1) and (3) of a rigid-body spacecraft has the following cascade structure, i.e., u controls ω and ω controls $\hat{q}$. This can be simply expressed as $u \rightarrow \omega \rightarrow \hat{q}$ or the following dynamic system

$$\dot{\hat{q}} = f(\hat{q}, \omega) \tag{4}$$

$$\dot{\omega} = g(\omega, u) \tag{5}$$

where

$$f(\hat{q}, \omega) = F(\omega)\hat{q}$$

$$F(\omega) = \frac{1}{2}\begin{bmatrix} 0 & \omega_3 & -\omega_2 & -\omega_1 \\ -\omega_3 & 0 & \omega_1 & -\omega_2 \\ \omega_2 & -\omega_1 & 0 & -\omega_3 \\ \omega_1 & \omega_2 & \omega_3 & 0 \end{bmatrix} \tag{6}$$

$$g(\omega, u) = J^{-1}(\Omega J\omega + u)$$

For the above cascade structure of the spacecraft dynamics we form the following problem.

Angular Velocity Shaping Problem: This problem is aimed to find the control law $u = u(\hat{q}, \omega)$ through the following two steps

1. **Find** *desired angular velocities* $\omega^* = h(\hat{q})$ such that when ω^* is taken as input to (4), the vector part q of the quaternion $\hat{q}$ will be regulated to zero and q_4 will reach to 1, i.e., the following nonlinear dynamical equations $\dot{\hat{q}} = f(\hat{q}, h(\hat{q}))$ will have $\hat{q}^f = [0,\ 0,\ 0,\ 1]^T$ as its asymptotically stable state.
2. **Shape** the actual angular velocity according to the desired angular velocity ω^* in the time domain, or make the actual angular velocity ω asymptotically track the desired angular velocity ω^*. From the asymptotically tracking condition

   $$\lim_{t\rightarrow\infty} ||\omega - \omega^*||_\infty = \lim_{t\rightarrow\infty} ||\omega - h(\hat{q})||_\infty = 0$$

   the final control law can be obtained as

   $$u_i = u_i(q, \omega),\ \ i = 1,2,3$$

   where $||\cdot||_\infty$ means the maximum absolute value of components of a vector.

□

A schematic diagram of this control logic is shown in Fig. 1.

In the following sections, the desired angular velocity is first obtained through quadratic feedback control law to stabilize the bilinear quaternion kinematics, then the final control logic is obtained through the asymptotically tracking condition.

3. Desired Angular Velocity

The time invariant bilinear system has the following state form

$$\dot{q}(t) = Aq(t) + B\omega(t) + \sum_{i=1}^{m} N_i q(t)\omega_i(t) \tag{7}$$

where $B = [b_1, b_2, \cdots, b_m] \in \mathcal{R}^{n\times m}$ and b_i $(i = 1, \cdots, m)$ are n-dimensional column vectors. The control input is $\omega = [\omega_1, \omega_2, \cdots, \omega_m]^T$.

The above system is linear in state $q(t)$ and input $\omega(t)$ but not jointly linear in q and ω. For this reason the elegant superposition rule doesn't apply to this system. Now we consider the quaternion kinematics (3). The quaternion $\hat{q}$ must be regulated to the final quaternion $\hat{q}^f = [0,\ 0,\ 0,\ 1]$ in the rest-to-rest sense. Define

$$\tilde{q} = \hat{q} - \hat{q}^f = [q_1,\ q_2,\ ,q_3,\ q_4 - 1]^T$$

Then the quaternion equation in (3) can be written as the following bilinear system with the angular velocity as input variables

$$\dot{\tilde{q}} = B\omega + N_1\tilde{q}\omega_1 + N_2\tilde{q}\omega_2 + N_3\tilde{q}\omega_3 \tag{8}$$

where

$$B = \frac{1}{2}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \frac{1}{2}[b_1,\ b_2,\ b_3]$$

$$N_1 = \frac{1}{2}\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}$$

$$N_2 = \frac{1}{2}\begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}$$

$$N_3 = \frac{1}{2}\begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

For the above bilinear quaternion equations (8), we have the following theorem.

Theorem 1: If we input any quadratic form of the following

$$\omega_i^* = -\alpha_i(N_i\tilde{q} + b_i)^T Q\tilde{q}, \quad i = 1,2,3 \tag{9}$$

to the bilinear quaternion kinematics (8), the resultant system will be globally and asymptotically stabilized. □

Remark: Since any positive definite symmetric matrix Q such that (9) will globally and asymptotically stabilize (8). Hence we can adjust the closed loop performance by choosing proper Q in (9). □

For different positive definite matrices Q's, (9) will give different quadratic feedback control law. For ease of consideration, we only consider the simple diagonal positive definite Q cases. We have the following desired angular velocities ω^*:

Desired angular velocity 1 ($Q = \text{diag}[1,\ 1,\ 1,\ 1]$):

$$\omega_i^*(q) = -\alpha_i q_i, \quad i = 1,2,3 \tag{10}$$

Desired angular velocity 2 ($Q = \text{diag}[2,\ 2,\ 2,\ 1]$):

$$\omega_i^*(\hat{q}) = -\alpha_i(1 + q_4)q_i, \quad i = 1,2,3 \tag{11}$$

Desired angular velocity 3 ($Q = \text{diag}[1,\ 1,\ ,1\ 2]$):

$$\omega_i^*(\hat{q}) = -\alpha_i(1 - q_4)q_i \tag{12}$$

The above three desired angular velocities can be expressed as the unified form

$$\omega_i^* = -\alpha_i k(q_4)q_i, \quad i = 1,2,3 \tag{13}$$

where $k(\cdot)$ is a function of $q_4(t)$

$$k(q_4) = \begin{cases} 1 & \text{for (12)} \\ 1 + q_4(t) & \text{for (13)} \\ 1 - q_4(t) & \text{for (14)} \end{cases}$$

Notice that both α_i and $k(q_4)$ in (13) are of positive values.

Denote the maximum slew rate as $\dot{\phi}_{\max}$, by choosing $\alpha_i = \dot{\phi}_{\max}$, the positive definite function

$$V_q(\hat{q}, t) = \frac{1}{2}\tilde{q}^T\tilde{q} = (1 - q_4) \tag{14}$$

will have the following time derivative along (3) closed by (13)

$$\dot{V}_q = -\frac{1}{2}\dot{\phi}_{\max}k(q_4)(1 + q_4)V_q < 0$$

or

$$V_q(\hat{q}, t) = V_q(\hat{q}(0), 0)\exp^{-\frac{1}{2}\dot{\phi}_{\max}\int_0^t k(q_4(\tau))(1+q_4(\tau))d\tau}$$

This means the parameter $\dot{\phi}_{\max}$ is related to the settling time of the quaternion regulation.

The above obtained desired angular velocities are proportional to q and the quadratic product of elements of $\hat{q}$. Since the magnitude of $\hat{q}$ is limited to equal to 1, the magnitude of the desired angular velocity will not exceed certain prescribed bounds $\dot{\phi}_{\max}$

$$\|\omega^*\|_2 = \sqrt{\omega^{*T}\omega^*} = \dot{\phi}_{\max}k(q_4)\sqrt{1 - q_4^2}$$

i.e.,the rate limitation requirement can be achieved by desired angular velocity for properly chosen α_i, $i = 1,2,3$.

Theorem 2: The quaternion will maneuver in the eigenaxis rotation sense [3] if the system angular velocity is kept in the desired angular velocity (9) with $\alpha_i = \dot{\phi}_{\max}$, $i = 1,2,3$. □

Proof: Substituting the ω^* in (9) into (3) leads to

$$\begin{aligned} \dot{q} &= -\frac{1}{2}\omega^* \times q + \frac{1}{2}q_4\omega^* \\ &= -\frac{1}{2}\dot{\phi}_{\max}k(q_4)q \times q - \frac{1}{2}\alpha k(q_4)q_4 q \\ &= -\frac{1}{2}\dot{\phi}_{\max}k(q_4)q_4 q \end{aligned}$$

i.e.

$$q(t) = c_q(t)q(0) = (\exp^{-\frac{1}{2}\dot{\phi}_{\max}\int_0^t q_4(\tau)k(q_4(\tau))d\tau})q(0) \tag{15}$$

This means eigenaxis rotation is achieved when inputing desired angular velocity (9) with identity α_i into (2). □

4. Tracking the Desired Angular Velocity

It is obvious that desired angular velocity ω^* can not be achieved by the actual angular velocity ω at the initial time for rest-to-rest maneuver. Since at the initial time, $q(0) \neq 0$. Although the desired angular velocity can not be achieved by the actual angular velocity at the very beginning of the maneuver, it is possible to make the actual angular velocity asymptotically approach the desired angular velocity. Whenever the actual angular velocity reaches the desired one, the maneuver will be eigenaxis rotation if the desired angular velocity is given in (9) with identical α_i.

The desired angular velocities should be followed by the actual angular velocities through control torque u, it is straight forward to form the following problem.

Angular velocity tracking problem: Find control vector u, such that the actual angular velocity asymptotically tracks the desired angular velocity ω^* and guarantee stability of the overall closed loop system

$$J\dot{\omega} = \Omega J\omega + u(q, \omega) \tag{16}$$

such that $\tilde{q}$ will be regulated to zero. □

The angular velocity tracking error $\tilde{\omega}$ is defined as

$$\tilde{\omega} = \omega - \omega^*$$

Consider the following 1st order angular velocity error dynamics

$$\dot{\tilde{\omega}} = -\lambda f(\tilde{\omega}) \tag{17}$$

where $f(\cdot)$ is any vector function of the form

$$f(\tilde{\omega}) = [f(\tilde{\omega}_1),\ f(\tilde{\omega}_2),\ f(\tilde{\omega}_3)]^T$$

with each of its component satisfying

$$x_i f(x_i) > 0,\ x_i \neq 0,\ \ f(0) = 0 \tag{18}$$

We choose the following Lyapunov function

$$V_\omega = \frac{1}{2}\tilde{\omega}^T\tilde{\omega} \tag{19}$$

The time derivative of V along the trajectory in (17) is

$$\dot{V}_\omega = -\lambda\tilde{\omega}^T f(\tilde{\omega}) \tag{20}$$

where λ is the positive scalar constant representing the tracking performance. It is obvious that $\dot{V}_\omega < 0$ and $\tilde{\omega} = \omega - \omega^* = 0$ is the only asymptotically stable equilibrium point of (17), or we can say the ω will asymptotically track ω^* when both of them satisfy the error dynamics (17).

By comparing (16) and (17), we obtain the control u such that the actual angular velocity tracks the desired angular velocity according to the dynamics (16) as follows

$$u = -\lambda J f(\tilde{\omega}) + J\dot{\omega}^* + \Omega J\omega \tag{21}$$

The above control law includes the differentiating term $\dot{\omega}^*$. This term actually is a function $\hat{q}$. Now we formulate $\dot{\omega}^*$. From last section, we obtain the desired angular velocity as

$$\omega^* = h(\hat{q}) = [h_1(\hat{q}),\ h_2(\hat{q}),\ h_3(\hat{q})]$$

hence the time derivation of ω^* could also be expressed as a function of $\hat{q}$

$$\dot{\omega}^* = h_d(\hat{q}) = H(\hat{q})\dot{\hat{q}} = H(\hat{q})F(\omega)\hat{q}$$

where $F(\omega)$ is same as in (6) and H is the Jacobian matrix of $h(\cdot)$ with respect to $\hat{q}$.

For the *desired angular velocity 1* (10), we have

$$H(\hat{q}) = \begin{bmatrix} -\alpha_1 & 0 & 0 & 0 \\ 0 & -\alpha_2 & 0 & 0 \\ 0 & 0 & -\alpha_3 & 0 \end{bmatrix}$$

For the *desired angular velocity 2* (11), we have

$$H(\hat{q}) = -\begin{bmatrix} \alpha_1(1+q_4) & 0 & 0 & \alpha_1 q_1 \\ 0 & \alpha_2(1+q_4) & 0 & \alpha_2 q_2 \\ 0 & 0 & \alpha_3(1+q_4) & \alpha_3 q_3 \end{bmatrix}$$

For the *desired angular velocity 3* (12), we have

$$H(\hat{q}) = -\begin{bmatrix} \alpha_1(1-q_4) & 0 & 0 & \alpha_1 q_1 \\ 0 & \alpha_2(1-q_4) & 0 & \alpha_2 q_2 \\ 0 & 0 & \alpha_3(1-q_4) & \alpha_3 q_3 \end{bmatrix}$$

The feedback control logic can be expressed as

$$u = -\lambda J f(\tilde{\omega}) + JH(\hat{q})F(\omega)\hat{q} + \Omega J\omega \tag{22}$$

There are several kinds of functions f's satisfying (18), we only consider the standard saturation function which is defined as

$$\text{sat}(\tilde{\omega}) = [\text{sat}(\tilde{\omega}_1),\ \text{sat}(\tilde{\omega}_2),\ \text{sat}(\tilde{\omega}_3)]$$

which satisfies

- $\tilde{\omega}_i\,\text{sat}(\tilde{\omega}_i) > 0$ for $\tilde{\omega}_i \neq 0$;
- $\text{sat}(\tilde{\omega}_i) = \tilde{\omega}_i$ for $|\tilde{\omega}_i| < a$;
- $|\text{sat}(\tilde{\omega}_i)| = a$ for $|\tilde{\omega}_i| \geq a$;

where a is properly chosen as the width of the boundary layer.

Then (22) has the the following form law

$$u = -\lambda J\,\text{sat}(\tilde{\omega}) + JH(\hat{q})F(\omega)\hat{q} + \Omega J\omega \tag{23}$$

The following theorem describe the robustness of (23).

Theorem 3: Consider the uncertain inertia matrix $J = J_0 + \Delta J$ with its nominal value J_0 and the norm-bounded perturbation ΔJ

$$\Delta J \in X = \{Y \mid \|Y\|_\infty \leq \delta\}$$

where δ is a given positive number.

If we use the controller (23) and choose the design parameter $\lambda > \lambda_0$ where

$$\lambda_0 = \dot{\phi}_{\max}(\dot{\phi}_{\max} + \frac{1}{2}\|H\|_\infty)\frac{\delta}{\sigma_{\min}(J_0)} \tag{24}$$

Then the system is robustly stable with respect to all $\Delta J \in X$. □

Proof: When $|\tilde{\omega}_i| \geq a$, the controller (23) can be expressed as

$$u = -\lambda J_0\,\text{sgn}(\tilde{\omega}) + J_0 H(\hat{q})F(\omega)\hat{q} - \Omega J_0\omega \tag{25}$$

Consider the matrix $F(\hat{q})$ defined in (6)

$$\|F\|_\infty = \sup_{t\in[0,+\infty)} \sigma_{\max}(F) \tag{26}$$

$$= \sup_{t\in[0,+\infty)} \frac{1}{2}\sqrt{\omega_1^2(t)+\omega_2^2(t)+\omega_3^2(t)} \leq \frac{1}{2}\dot{\phi}_{\max}$$

where $\sigma_{\max}(\cdot)$ means the maximum singular value of a matrix. Also consider

$$F(\omega) = \frac{1}{2}\begin{bmatrix} \Omega & -\omega \\ \omega^T & 0 \end{bmatrix}$$

from the property of singular value (the singular value of any minor of a matrix is less than the singular value of the matrix itself), we obtain

$$\|\Omega\|_\infty \leq 2\|F\|_\infty = \dot{\phi}_{\max}$$

Consider the mismatched error dynamics

$$(J_0 + \Delta J)\dot{\tilde{\omega}} = -\lambda J_0\,\text{sgn}(\tilde{\omega}) + \Omega\Delta J\omega - \Delta JH(\hat{q})F(\omega)\hat{q}$$

Since $J_0 + \Delta J$ is always positive definite, we choose the following positive definite function

$$V_\omega = \frac{1}{2}\tilde{\omega}^T(J_0 + \Delta J)\tilde{\omega}$$

which has time derivative

$$\dot{V}_\omega = -\lambda J_0\tilde{\omega}^T\text{sgn}(\tilde{\omega}) + \tilde{\omega}^T\Omega\Delta J\omega - \tilde{\omega}^T\Delta JH(\hat{q})F(\omega)\hat{q}$$

If we choose $\lambda > \lambda_0$, then we have

$$\lambda \geq \|J_0^{-1}\Omega\Delta J\omega\|_\infty + \|J_0^{-1}\Delta JH(\hat{q})F(\omega)\hat{q}\|_\infty$$

which implies $\dot{V}_\omega < 0$

When $|\tilde{\omega}_i| < a$, i.e., $\tilde{\omega}$ lies within the linear range of the saturation function, $\dot{V}_\omega$ may be negative or positive. We don't need worry about negative case. For positive $\dot{V}_\omega$ case, $\tilde{\omega}$ will increase until it lies outside the linear range. Whenever $\tilde{\omega}$ reaches the saturation range, $\dot{V}_\omega < 0$. Hence the closed loop system is Lyapunov stable. If the boundary layer width a is chosen to be sufficiently small such that it is within settling-down range of $\tilde{\omega}$, then the Lyapunov stability can be thought as asymptotical stability. □

5. Stability and Performance Analysis

In order to properly choose the design parameters, we need to study how the design parameters affect the closed loop performance. In order to make the actual angular velocity shape the desired angular velocity in time domain, hence the time-response performance of the quaternion regulation and the angular velocity tracking have to be studied.

The design parameters considered here are quaternion settling-time parameter α, angular velocity tracking error settling-time parameter λ and the boundary layer width a. Since the settling time concept of the step time response of a closed loop system is will-defined, but the settling time for the regulation is not available from any textbook. For this reason we give the following settling-time definition of the regulation version.

Definition: The settling time of a regulated system is called the *regulation settling-time*, which is defined as the time required for the time-response curve to reach and stay within a range around zero of size specified by absolute percentage of the initial value (usually 2% or 5%). □

For this regulation settling-time, the 4-times-time-constant criterion may not valid. Consider the following example

$$x(t) = 2\exp^{-t/\tau}x(0),\quad x(4.6\tau) = 0.02x(0)$$

where τ is time constant. Hence 2%-settling-time of $x(t)$ is $t = 4.6\tau$ second, i.e., the 4-factor in step response should be replaced by 4.6-factor in this regulation response.

1. Settling-time t_q of $\hat{q}$ under ω^*: Substituting (13) into the quaternion kinematics equations (3), we obtain the differential equation for $q_4(t)$

$$\dot{q}_4 = \frac{1}{2}\alpha k(q_4)q^Tq = \frac{1}{2}\alpha k(q_4)(1-q_4^2) \tag{27}$$

this is a first order differential equation and can be easily solved for q_4. After finding $q_4(t)$, from (15), $q(t)$ can be obtained analytically.

For $k(q_4) = 1$, i.e., the desired angular velocity is taken as (10), we have the time response of $\hat{q}(t)$ expressed as follows

$$q_4(t) = 1 - 2c_1\frac{\exp^{-\alpha t}}{1 + c_1\exp^{-\alpha t}} \tag{28a}$$

$$q(t) = \frac{1+c_1}{1+c_1\exp^{-\alpha t}}\exp^{-0.5\alpha t}q(0) \tag{28b}$$

where

$$c_1 = \frac{1-q_4(0)}{1+q_4(0)}$$

Since $0 \leq q_4(0) \leq 1$, hence

$$\frac{1+c_1}{1+c_1\exp^{-\alpha t}} \leq 2$$

It is obvious that the settling-time in this case is

$$t_q = \frac{4.6}{0.5\alpha} = \frac{9.2}{\alpha} \tag{29}$$

2. Settling time t_ω of tracking error dynamics: Consider the following dynamics

$$\dot{\tilde{\omega}} = -\lambda\,\text{sat}(\tilde{\omega})$$

the saturation function has the boundary layer width a. If the boundary layer a is sufficiently small, then whenever $\tilde{\omega}$ reaches the linear range of the saturation, we think $\tilde{\omega}$ is settled down in the sense of regulation settling time.

Since at the initial time the angular velocity tracking error

$$\tilde{\omega}(0) = -\omega^*(0) = \alpha q(0)$$

is large, we assume the saturation will last from $t = 0$ to $t = t_1$ and ends at time t_1, i.e.

$$\pm a = -\omega_i^*(0) + \lambda a\,\text{sgn}(\omega_i^*(0))t_1$$

hence the possible maximum value of t_1 is

$$t_1 = \frac{a + \alpha\|q(0)\|_\infty}{\lambda a} \quad (30)$$

Considering small a such that the boundary layer range is same as the settling-down range of $\tilde{\omega}$, then the settling time t_ω is equal to t_1. Considering $\|q(0)\|_\infty \leq 1$, hence the settling time of the saturated angular tracking dynamics can be approximately expressed as

$$t_\omega = \frac{a + \alpha}{\lambda a} \quad (31)$$

Theorem 4: The actual angular velocity ω will never exceed the peak value of ω^* if we use the control law (23) for rest-to-rest maneuver. □

Proof: The actual angular velocity tries to track ω^* from $\omega(0) = 0$ and $\omega^*(0) \neq 0$. Hence at the very beginning, we have

$$|\omega_i(0)| < |\omega_i^*(0)|$$

Before $\tilde{\omega}$ is settled down or for $0 \leq t \leq t_\omega$

$$\omega = \omega^* + \lambda at\,\mathrm{sgn}(\omega - \omega^*) = \omega^* - \lambda at\,\mathrm{sgn}(\omega^*)$$

this means

$$|\omega_i(t)| < |\omega_i^*(t)| \;\text{ for } 0 \leq t \leq t_\omega$$

For $t > t_\omega$, we have

$$|\omega_i - \omega_i^*| \leq a \text{ or } |\omega_i| \leq |\omega_i^*| + a$$

Since the peak of ω^* occurs at $t = 0$ and $\omega^*(t)$ is exponentially decreasing, and a is sufficiently small, hence for $t > t_\omega$

$$|\omega_i^*(t)| + a < |\omega_i^*(0)|$$

i.e., $|\omega_i(t)| < \max|\omega_i^*(t)| = |\omega_i^*(0)|$. □

Theorem 5: The *settling time condition* of follows

$$t_\omega < t_q \quad (32)$$

will guarantee the overall stability of the rigid-body dynamics (1) closed by the control law (22). □

Proof: By substituting (22) to (1), we obtain the following closed loop system

$$\dot{\tilde{\omega}} = -\lambda f(\tilde{\omega}) \quad (33)$$

$$\dot{q} = \frac{1}{2}\Omega q + \frac{1}{2}q_4\omega \quad (34)$$

$$\dot{q}_4 = -\frac{1}{2}\omega^T q \quad (35)$$

Since from (20) the time derivative of V_ω satisfies

$$\dot{V}_\omega < 0, \; i.e., \; \lim_{t\to\infty} \tilde{\omega} = 0$$

or there exists a small positive number ϵ_1 such that

$$\tilde{\omega}^T(t)\tilde{\omega}(t) < \epsilon_1, \text{ whenever } t > t_\omega$$

The settling time condition (32) tells us that the regulation of $\tilde{\omega}$ should be faster than the regulation of q, i.e., after time instant t_ω, q is not settled down until t reaches t_q. Hence there exists another small positive number ϵ such that

$$\tilde{\omega}^T(t)\tilde{\omega}(t) < \epsilon q(t)^T q(t), \text{ whenever } t \geq t_\omega$$

Consider now the positive definite function V_q in (14). The time derivative of V_q is

$$\dot{V}_q = -2\dot{q}_4 = \omega^T q$$

From theorem 4, we have the fact that the magnitude of ω never exceeds the magnitude of the desired angular velocity, i.e., ω is bounded. Also q is bounded. Hence we know $\dot{V}_q(t)$ is bounded for all t.

On the other hand

$$\dot{V}_q = \omega^{*T} q + \tilde{\omega}^T q$$

The Schwarz inequality leads

$$(\tilde{\omega}^T q)^2 < (q^T q)(\tilde{\omega}^T\tilde{\omega})$$

If we choose $\alpha > \sqrt{\epsilon}$, then

$$\dot{V}_q \leq -(\alpha - \sqrt{\epsilon})q^T q < 0, \text{ for } t \geq t_\omega$$

this means

$$\lim_{t\to\infty} q = 0$$

Hence, finally we have

$$\lim_{t\to\infty} \omega = \lim_{t\to\infty} \omega^* = -\alpha \lim_{t\to\infty} q = 0$$

i.e., the closed loop system is stable and the control logic (23) regulates q and ω to zero. □

6. Design Procedure

The nonlinear quaternion feedback controller proposed in this paper has the following design parameters: the parameter α which determines the settling-time for the regulation of q; the angular velocity tracking parameter λ which determines the settling time of the angular velocity tracking error; the width of the boundary layer a of the saturation function.

In this section we summary the procedure how to pick up these design parameters to satisfy the performance requirement and the limitation requirements.

1. Pick up design parameter α:

For a given maximum bound $\dot{\phi}_{\max}$ of the slewing rate $\sqrt{\omega_1^2 + \omega_2^2 + \omega_3^2}$, we simply pick $\alpha = \dot{\phi}_{\max}$ then the actual angular velocity will never exceed this bound $\dot{\phi}_{\max}$ if we choose the desired angular velocity (10).

For the desired angular velocity (10) case, (27) shows that $q_4(t)$ will always satisfies $q_4(0) \leq q_4(t) \leq 1$, hence the magnitude of the actual angular velocity will satisfy

$$\|\omega\|_2 \leq \|\omega^*\|_2 \leq \alpha\sqrt{1 - q_4^2(0)}$$

This means if we choose

$$\alpha = \frac{\dot{\phi}_{\max}}{\sqrt{1 - q_4^2(0)}}$$

the controller will be less conservative for the rate limitation.

2. Pick up boundary layer width a:

For the purpose of making the boundary layer range of the saturation function be the angular velocity error settling-down range, we choose $a = 2\%\dot{\phi}_{max}$

3. Pick up angular velocity error parameter λ:

The successful control logic as shown in theorem 5 requires that the settling time of the angular velocity tracking error has to be less than the settling time of the quaternion's regulation. If we choose $t_q = 4t_\omega$, then we have

$$\lambda = \frac{\alpha(a + \alpha)}{2.3a} \quad (36)$$

4. Boundedness of control effort:

Now we estimate the bound for control vector u. Consider the desired angular velocity 1 in (10), the control (23) can be expressed as for $\alpha_i = \dot{\phi}_{\max}$

$$u = -\lambda J\,\mathrm{sat}(\tilde{\omega}) - \alpha JF(\omega)\hat{q} - \Omega J\omega$$

For diagonal J case, we have

$$\Omega J\omega = \begin{bmatrix} (J_3 - J_2)\omega_2\omega_3 \\ (J_1 - J_3)\omega_3\omega_1 \\ (J_2 - J_1)\omega_1\omega_2 \end{bmatrix} \quad (37)$$

Since we need enforce $\omega^T\omega \leq \dot{\phi}_{\max}^2$, hence (37) implies

$$\|\Omega J\omega\|_\infty \leq \frac{1}{2}J_d\dot{\phi}_{\max}^2$$

where

$$J_d = \max\{|J_3 - J_2|, |J_1 - J_3|, |J_2 - J_1|\}$$

Considering (26) and (36), then

$$\|u\|_\infty < \bar{u}$$

where

$$\bar{u} = (\frac{217}{115} + \frac{J_d}{J_{\max}})\frac{1}{2}J_{\max}\dot{\phi}_{\max}^2 \quad (38)$$

where

$$J_{\max} = \max_i J_i$$

If given maximum slew rate $\dot{\phi}_{\max}$, the upper bound of the control can be obtained from (38). If given upper bound of control effort $\bar{u}$, the allowed maximum slew rate can be obtained as

$$\dot{\phi}_{\max} = \sqrt{\frac{2\bar{u}}{1.887J_{\max} + J_d}} \quad (39)$$

In the case of both $\bar{u}$ and $\dot{\phi}_{\max}$ are given, if we want to fix $\bar{u}$ then we can choose

$$\dot{\phi}_{\max}^{\text{new}} = \min\{\dot{\phi}_{\max}, \sqrt{\frac{2\bar{u}}{1.887J_{\max} + J_d}}\}$$

if we want to fix $\dot{\phi}_{\max}$, we can choose

$$\bar{u}^{\text{new}} = \min\{\bar{u}, (1.887 + \frac{J_d}{J_{\max}})\frac{1}{2}J_{\max}\dot{\phi}_{\max}^2\}$$

7. Example

We consider X-ray Timing Explorer (XTE) [7] satellite as an example to illustrate the previous procedure for controller design. XTE satellite is the next in a long series of Explorer-class missions developed by NASA. It will study the structure and dynamics of compact x-ray sources, neutron stars, white dwarfs, and other stellar objectives with x-ray energy emission.

The maximum control bound is given as

$$\bar{u} = 0.4 \ \text{n·m}$$

The limitation on the slewing rate is

$$\dot{\phi}_{max} = 0.01 \ \text{rad/sec}$$

The inertia matrix of XTE is

$$J = \text{diag}(6292,\ 5477,\ 2687)$$

in units of kg·m^2.

The initial quaternion are

$$\bar{q}(0) = [0.2652,\ 0.2652,\ -0.6930,\ 0.6157]^T$$

We take the design parameter

$$\alpha_i = \frac{\dot{\phi}_{max}}{\sqrt{1 - q_4^2(0)}} = 0.0127,\ i = 1, 2, 3$$

and the boundary layer width a of the saturation function is taken as the settling-down rage, i.e.,

$$a = 2\%\alpha = 0.2538 \times 10^{-3}$$

From theorem 5 we choose $t_q = 4t_\omega$, then

$$\lambda = 0.2814$$

The time histories of u, ω, $\dot{\omega}$ and q, are shown in Fig. 2 and Fig. 3.

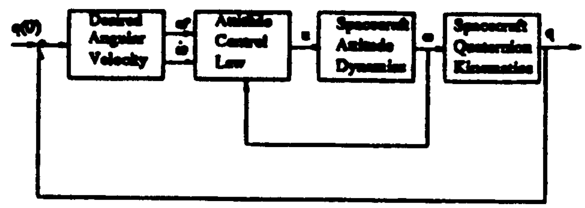

Figure 1: Block diagram of the controlled system

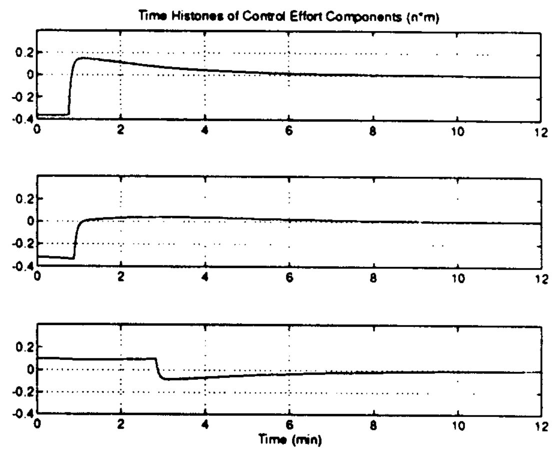

Figure 2: Time histories of control effort u.

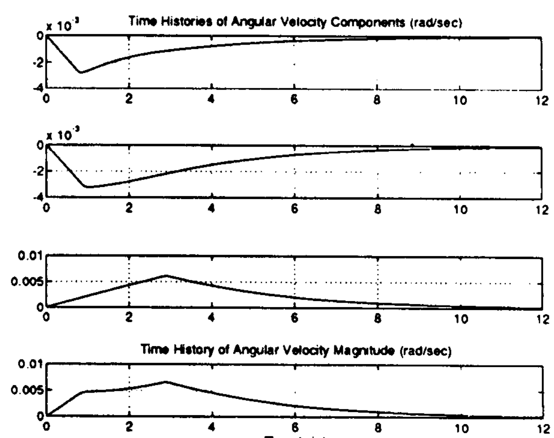

Figure 3: Time histories of angular velocity ω.

VIII. Conclusion

This paper presents a new attitude control law for rest-to-rest maneuver of a spacecraft. This new approach is called *angular velocity shaping method.* The desired angular velocity, which is tracked by the actual angular velocity, is different from the command angular velocity. The latter is an open loop strategy. By using this method we can directly limit the slewing rate to any prescribed bound. The control law is very simple and can be easily implemented and it guarantee robustness in the maneuvering.

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